STARLIKENESS AND CONVEXITY OF ORDER $\alpha$ AND TYPE $\beta$ FOR P-VALENT HYPERGEOMETRIC FUNCTIONS

R. M. EL-ASHWAH

Abstract. Given the hypergeometric function $F(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n(1)_n} z^n$, we place conditions on $a, b, c$ to guarantee that $z^p F(a, b; c; z)$ will be in various subclasses of $p$-valent starlike and $p$-valent convex functions of order $\alpha$ and type $\beta$ ($0 \leq \alpha < p, 0 < \beta \leq 1$). Operators related to the hypergeometric function are also examined.

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1. INTRODUCTION

Let $S(p)$ be the class of functions of the form:

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n \quad (p \in \mathbb{N} = \{1, 2, \ldots\})$$  \hspace{1cm} (1.1)

which are analytic and $p$-valent in the unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$. A function $f(z) \in S(p)$ is called $p$-valent starlike of order $\alpha$ if $f(z)$ satisfies

$$\text{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} > \alpha \hspace{1cm} (1.2)$$

for $0 \leq \alpha < p, p \in \mathbb{N}$ and $z \in U$. We denote by $S_p^*(\alpha)$ the class of all $p$-valent starlike functions of order $\alpha$ and $S_p^*(0) = S_p^*$. Denote by $S_p^*(\alpha, \beta)$ the subclass consisting of functions $f(z) \in S(p)$ which satisfy

$$\left| \frac{zf'(z)}{f(z)} - p \right| < \beta \hspace{1cm} (1.3)$$

for $0 \leq \alpha < p, 0 < \beta \leq 1, p \in \mathbb{N}$ and $z \in U$. Also a function $f(z) \in S(p)$ is called $p$-valent convex of order $\alpha$ if $f(z)$ satisfies
\[ \text{Re} \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > \alpha \] 

for \( 0 \leq \alpha < p, \, p \in \mathbb{N} \) and \( z \in U \). We denote by \( K_p(\alpha) \) the class of all \( p \)-valent convex functions of order \( \alpha \) and \( K_p(0) = K_p \). Also denote by \( K_p(\alpha, \beta) \) the subclass consisting of functions \( f(z) \in S(p) \) which satisfy

\[ \left| \frac{1 + \frac{zf''(z)}{f'(z)} - p}{1 + \frac{zf''(z)}{f'(z)} + p - 2\alpha} \right| < \beta \] 

for \( 0 \leq \alpha < p, 0 < \beta \leq 1, \, p \in \mathbb{N} \) and \( z \in U \).

It follows from (1.3) and (1.5) that

\[ f(z) \in K_p(\alpha, \beta) \iff \frac{zf'(z)}{p} \in S_p(\alpha, \beta). \] 

Denoting by \( T(p) \) the subclass of \( S(p) \) consisting of functions of the form:

\[ f(z) = z^p - \sum_{n=p+1}^{\infty} a_n z^n \quad (a_{p+n} \geq 0; \, p \in \mathbb{N}). \] 

We denote by \( T_p^*(\alpha), T_p^*(\alpha, \beta), C_p(\alpha) \) and \( C_p(\alpha, \beta) \) the classes obtained by taking intersections, respectively, of the classes \( S_p^*(\alpha), S_p^*(\alpha, \beta), K_p(\alpha) \) and \( K_p(\alpha, \beta) \) with the class \( T(p) \)

\[ T_p^* = S_p^* \cap T(p) \]
\[ T_p^*(\alpha) = S_p^*(\alpha) \cap T(p) \]
\[ T_p^*(\alpha, \beta) = S_p^*(\alpha, \beta) \cap T(p) \]
\[ C_p = K_p \cap T(p) \]
\[ C_p(\alpha) = K_p(\alpha) \cap T(p) \]

and

\[ C_p(\alpha, \beta) = K_p(\alpha, \beta) \cap T(p). \]
The class $S_p^*(\alpha)$ was studied by Patil and Thakare [8]. The classes $T_p^*(\alpha)$ and $C_p(\alpha)$ were studied by Owa [7], and the classes $T_p^*(\alpha, \beta)$ and $C_p(\alpha, \beta)$ were studied by Hossen [4] (see also [1]).

For $a, b, c \in \mathbb{C}$ and $c \neq 0, -1, -2, \ldots$, the Gaussian hypergeometric function is defined by:

$$F(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n (1)_n} z^n \quad (z \in U), \quad (1.8)$$

where $(\lambda)_n$ is the Pochhammer symbol defined, in terms of the Gamma function $\Gamma$, by

$$(\lambda)_n = \frac{\Gamma(\lambda + n)}{\Gamma(\lambda)} = \left\{ \begin{array}{ll}
1 & (n = 0) \\
\lambda (\lambda + 1) \ldots \ldots (\lambda + n - 1) & (n \in \mathbb{N}).
\end{array} \right. \quad (1.9)$$

The series in (1.8) represents an analytic function in $U$ and has an analytic continuation throughout the finite complex plane except at most for the cut $[1, \infty)$. We note that $F(a, b; c; 1)$ converges for $\Re(c - a - b) > 0$ and is related to the Gamma function by

$$F(a, b; c; 1) = \frac{\Gamma(c)\Gamma(c - a - b)}{\Gamma(c - a)\Gamma(c - b)}. \quad (1.10)$$

Corresponding to the function $F(a, b; c; z)$ we define

$$h_p(a, b; c; z) = z^p F(a, b; c; z). \quad (1.11)$$

We observe that, for a function $f(z)$ of the form (1.1), we have

$$h_p(a, b; c; z) = z^p + \sum_{n=p+1}^{\infty} \frac{(a)_{n-p} (b)_{n-p}}{(c)_{n-p} (1)_{n-p}} z^n. \quad (1.12)$$

In [3] El-Ashwah et al. gave necessary and sufficient conditions for $z^p F(a, b; c; z)$ to be in the classes $T_p^*(\alpha)$ and $C_p(\alpha)$ ($0 \leq \alpha < p$) and has also examined a linear operator acting on hypergeometric functions. Also in [10] Silverman gave necessary and sufficient conditions for $z F(a, b; c; z)$ to be in the classes $T_1^*(\alpha) = T^*(\alpha)$ and $C_1(\alpha) = C(\alpha)$ ($0 \leq \alpha < 1$) and has also examined a linear operator acting on hypergeometric functions. Also in [6] Mostafa obtained analogous results for the classes $S^*(\alpha, \beta)$ and $C(\alpha, \beta)$ ($0 \leq \alpha < 1, 0 < \beta \leq 1$). For the other interesting developments for $z F(a, b; c; z)$ in connection with various subclasses of univalent functions, the reader can refer to the works of Carlson and Shaffer [2], Merkes and Scott [5] and Ruscheweyh and Singh [9].

In the present paper, we determine necessary and sufficient conditions for $h_p(a, b; c; z)$ to be in the classes $T_p^*(\alpha, \beta)$ and $C_p(\alpha, \beta)$.

Furthermore, we consider an integral operator related to the hypergeometric function.
2. Main Results

To establish our main results, we shall need the following lemmas.

**Lemma 1 [4].** Let the function \( f(z) \) defined by (1.1).

(i) A sufficient condition for \( f(z) \in S(p) \) to be in the class \( S^*_p(\alpha, \beta) \) is that

\[
\sum_{n=p+1}^{\infty} \left\{ n(1 + \beta) - \left[ p(1 - \beta) + 2\alpha\beta \right] \right\} |a_n| \leq 2\beta(p - \alpha).
\]

(ii) A sufficient condition for \( f(z) \in S(p) \) to be in the class \( K^*_p(\alpha, \beta) \) is that

\[
\sum_{n=p+1}^{\infty} \frac{n}{p} \left\{ n(1 + \beta) - \left[ p(1 - \beta) + 2\alpha\beta \right] \right\} |a_n| \leq 2\beta(p - \alpha).
\]

**Lemma 2 [4].** Let the function \( f(z) \) defined by (1.7). Then

(i) \( f(z) \in T(p) \) is in the class \( T^*_p(\alpha, \beta) \) if and only if

\[
\sum_{n=p+1}^{\infty} \left\{ n(1 + \beta) - \left[ p(1 - \beta) + 2\alpha\beta \right] \right\} a_n \leq 2\beta(p - \alpha).
\]

(ii) \( f(z) \in T(p) \) is in the class \( C^*_p(\alpha, \beta) \) if and only if

\[
\sum_{n=p+1}^{\infty} \frac{n}{p} \left\{ n(1 + \beta) - \left[ p(1 - \beta) + 2\alpha\beta \right] \right\} a_n \leq 2\beta(p - \alpha).
\]

**Theorem 1.** If \( a, b > 0 \) and \( c > a + b + 1 \), then a sufficient condition for \( h_p(a, b; c; z) \) to be in the class \( S^*_p(\alpha, \beta) \) (\( 0 \leq \alpha < p, 0 < \beta \leq 1 \)) is that

\[
\frac{\Gamma(c)\Gamma(c - a - b)}{\Gamma(c - a)\Gamma(c - b)} \left[ 1 + \frac{ab(1 + \beta)}{2\beta(p - \alpha)(c - a - b - 1)} \right] \leq 2. \tag{2.1}
\]

Condition (2.1) is necessary and sufficient for \( F_p \) defined by \( F_p(a, b; c; z) = z^p(2 - F(a, b; c; z)) \) to be in the class \( T^*_p(\alpha, \beta) \).

**Proof.** Since \( h_p(a, b; c; z) = z^p + \sum_{n=p+1}^{\infty} \frac{(a)_{n-p}(b)_{n-p}}{(c)_{n-p}(1)_{n-p}} z^n \), according to Lemma 1 (i), we need only show that

\[
\sum_{n=p+1}^{\infty} \left\{ n(1 + \beta) - \left[ p(1 - \beta) + 2\alpha\beta \right] \right\} \frac{(a)_{n-p}(b)_{n-p}}{(c)_{n-p}(1)_{n-p}} a_n \leq 2\beta(p - \alpha).
\]
Now
\[ \sum_{n=p+1}^{\infty} \{ n(1+\beta) - [p(1-\beta) + 2\alpha\beta] \} \frac{(a)_{n-p}(b)_{n-p}}{(c)_{n-p}(1)_{n-p}} \]
\[ = \sum_{n=1}^{\infty} \{ (n+p)(1+\beta) - [p(1-\beta) + 2\alpha\beta] \} \frac{(a)_{n}(b)_{n}}{(c)(1)_{n}} \]
\[ = (1+\beta) \sum_{n=1}^{\infty} \frac{(a)_{n}(b)_{n}}{(c)(1)_{n}} + 2\beta(p-\alpha) \sum_{n=1}^{\infty} \frac{(a)_{n}(b)_{n}}{(c)(1)_{n}}. \tag{2.2} \]

Noting that \((\lambda)_{n} = \lambda(\lambda+1)_{n-1}\) and then applying (1.10), we may express (2.2) as
\[ \frac{ab}{c} (1+\beta) \sum_{n=1}^{\infty} \frac{(a+1)_{n-1}(b+1)_{n-1}}{(c+1)_{n-1}(1)_{n-1}} + 2\beta(p-\alpha) \sum_{n=1}^{\infty} \frac{(a)_{n}(b)_{n}}{(c)(1)_{n}} \]
\[ = \frac{ab}{c} (1+\beta) \frac{\Gamma(c+1)\Gamma(c-a-b-1)}{\Gamma(c-a)\Gamma(c-b)} + 2\beta(p-\alpha) \left[ \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} - 1 \right] \]
\[ = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \left[ \frac{ab(1+\beta)}{c-a-b-1} + 2\beta(p-\alpha) \right] - 2\beta(p-\alpha) \]

But this last expression is bounded above by \(2\beta(p-\alpha)\) if and only if (2.1) holds.

Since \(F_{p}(a,b;c;z) = z^{p} - \sum_{n=p+1}^{\infty} \frac{(a)_{n-p}(b)_{n-p}}{(c)_{n-p}(1)_{n-p}} z^{n}\), the necessity of (2.1) for \(F_{p}\) to be in the class \(T_{p}^{*}(\alpha,\beta)\) follows from Lemma 2 (i).

In the next theorem, we find constraints on \(a, b\) and \(c\) that lead to necessary and sufficient conditions for \(h_{p}(a,b;c;z)\) to be in the class \(T_{p}^{*}(\alpha,\beta)\).

**Theorem 2.** If \(a,b > -1, c > 0, \text{ and } ab < 0\), then a necessary and sufficient condition for \(h_{p}(a,b;c;z)\) to be in the class \(T_{p}^{*}(\alpha,\beta)\) is that \(c \geq a + b + 1 - \frac{ab(1+\beta)}{2\beta(p-\alpha)}\).

The condition \(c \geq a + b + 1 - \frac{ab}{p}\) is necessary and sufficient for \(h_{p}(a,b;c;z)\) to be in the class \(T_{p}^{*}\).

**Proof.** Since
\[ h_{p}(a,b;c;z) = z^{p} + \sum_{n=p+1}^{\infty} \frac{(a)_{n-p}(b)_{n-p}}{(c)_{n-p}(1)_{n-p}} z^{n} \]
\[ = z^{p} + \frac{ab}{c} \sum_{n=p+1}^{\infty} \frac{(a+1)_{n-p-1}(b+1)_{n-p-1}}{(c+1)_{n-p-1}(1)_{n-p}} z^{n} \]
\[ \sum_{n=p+1}^{\infty} \{n(1+\beta) - [p(1-\beta) - 2\alpha\beta]\} \frac{(a+1)n(b+1)}{(c+1)n(1)n+1} \leq \left| \frac{c}{ab} \right| 2\beta(p-\alpha). \]  

Note that the left side of (2.4) diverges if \( c \leq a + b + 1 \). Now
\[ \sum_{n=0}^{\infty} \{(n+p+1)(1+\beta) - [p(1-\beta) - 2\alpha\beta]\} \frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_n+1} \]
\[ = (1+\beta) \sum_{n=0}^{\infty} \frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_n} + 2\beta(p-\alpha) \frac{c}{ab} \sum_{n=1}^{\infty} \frac{(a)_n(b)_n}{(c)_n(1)_n} \]
\[ = \frac{\Gamma(c+1)\Gamma(c-a-b-1)}{\Gamma(c-a)\Gamma(c-b)} (1+\beta) + 2\beta(p-\alpha) \frac{c}{ab} \left[ \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} - 1 \right]. \]

Hence, (2.4) is equivalent to
\[ \frac{\Gamma(c+1)\Gamma(c-a-b-1)}{\Gamma(c-a)\Gamma(c-b)} \left[ (1+\beta) + 2\beta(p-\alpha) \frac{c-a-b-1}{ab} \right] \leq 2\beta(p-\alpha) \left[ \frac{c}{|ab|} + \frac{c}{ab} \right] = 0. \]  

Thus, (2.5) is valid if and only if
\[ (1+\beta) + 2\beta(p-\alpha) \frac{(c-a-b-1)}{ab} \leq 0, \]
or, equivalently,
\[ c \geq a + b + 1 - \frac{ab(1+\beta)}{2\beta(p-\alpha)}. \]

Another application of Lemma 2 (i) when \( \alpha = 0 \) and \( \beta = 1 \) completes the proof of Theorem 2.

Our next theorems will parallel Theorems 1 and 2 for the p-valent convex case.

**Theorem 3.** If \( a, b > 0 \) and \( c > a + b + 2 \), then a sufficient condition for \( h_p(a, b; c; z) \) to be in the class \( K_p(\alpha, \beta) \), \( 0 \leq \alpha < p, 0 < \beta \leq 1 \), is that
\[ \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \left[ 1 + \frac{(2p+1)(1+\beta) - [p(1-\beta) + 2\alpha\beta]}{2\beta p(p-\alpha)} \frac{ab}{(c-a-b-1)} + \right. \]
\[ \frac{(1 + \beta)(a_2(b)_2)}{2\beta p(p - \alpha)(c - a - b - 2)_2} \leq 2. \quad (2.6) \]

Condition (2.6) is necessary and sufficient for \( F_p(a, b; c; z) = z^p(2 - F(a, b; c; z)) \) to be in the class \( C_p(\alpha, \beta) \).

**Proof.** In view of Lemma 1 (ii), we need only show that
\[
\sum_{n=p+1}^{\infty} n \{ n(1 + \beta) - [p(1 - \beta) + 2\alpha \beta] \} \frac{(a)_{n-p}(b)_{n-p}}{(c)_{n-p}(1)_{n-p}} \leq 2\beta p(p - \alpha).
\]

Now
\[
\sum_{n=0}^{\infty} (n + p + 1) \{ (n + p + 1)(1 + \beta) - [p(1 - \beta) + 2\alpha \beta] \} \frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_{n+1}}
\]
\[
= (1 + \beta) \sum_{n=0}^{\infty} (n + 1)^2 \frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_{n+1}} + 2\beta p(1 + \beta) - [p(1 - \beta) + 2\alpha \beta] \sum_{n=0}^{\infty} (n + 1) \frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_{n+1}}
\]
\[
= (1 + \beta) \sum_{n=0}^{\infty} (n + 1) \frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_{n+1}} + (2 p + 1)(1 + \beta) - [p(1 - \beta) + 2\alpha \beta] \sum_{n=0}^{\infty} (a)_{n+1}(b)_{n+1}
\]
\[
= (1 + \beta) \sum_{n=1}^{\infty} \frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_{n+1}} + (2 p + 1)(1 + \beta) - [p(1 - \beta) + 2\alpha \beta] \sum_{n=0}^{\infty} \frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_{n+1}}
\]
\[
= (1 + \beta) \sum_{n=0}^{\infty} \frac{(a)_{n+2}(b)_{n+2}}{(c)_{n+2}(1)_{n}} + (2 p + 1)(1 + \beta) - [p(1 - \beta) + 2\alpha \beta] \sum_{n=0}^{\infty} \frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_{n}}
\]
\[
+ 2\beta p(p - \alpha) \sum_{n=0}^{\infty} \frac{(a)_{n}(b)_{n}}{(c)_{n}(1)_{n}}. \quad (2.7)
\]

Since \((a)_{n+k} = (a)_k(a + k)_{n}\), we may write (2.7) as
\[
\frac{(a)_{2}(b)_{2}}{(c)_{2}} \frac{\Gamma(c + 2)\Gamma(c - a - b - 2)}{\Gamma(c - a)\Gamma(c - b)} (1 + \beta) + \{(2 p + 1)(1 + \beta) - [p(1 - \beta) + 2\alpha \beta]\} \frac{ab}{c}.
\]
We have
\[
\frac{\Gamma(c+1)\Gamma(c-a-b-1)}{\Gamma(c-a)\Gamma(c-b)} + 2\beta p(p-\alpha) \left[ \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} - 1 \right].
\]

Upon simplification, we see that this last expression is bounded above by \(2\beta p(p-\alpha)\) if and only if \((2.6)\) holds. That \((2.6)\) is also necessary for \(F_p\) to be in the class \(C_p(\alpha, \beta)\) follows from Lemma 2 (ii).

**Theorem 4.** If \(a, b > -1, ab < 0\) and \(c > a + b + 2\), then a necessary and sufficient condition for \(h_p(a, b; c; z)\) to be in the class \(C_p(\alpha, \beta)\) is that

\[
(a)_{2b}(1+\beta) + \{(2p+1)(1+\beta) - [p(1-\beta) + 2\alpha \beta]\} ab(c-a-b-2)
+ 2\beta p(p-\alpha)(c-a-b-2) \geq 0. 
\]

*(2.8)*

**Proof.** Since \(h_p(a, b; c; z)\) has the form (2.3), we see from Lemma 2 (ii) that our conclusion is equivalent to

\[
\sum_{n=p+1}^{\infty} n \{n(1+\beta) - [p(1-\beta) + 2\alpha \beta]\} \frac{(a+1)n-b+11n-p-1}{(c+1)n-p-1(1)n-p} \leq \left| \frac{c}{ab} \right| 2\beta p(p-\alpha).
\]

*(2.9)*

Note that \(c > a + b + 2\) if the left hand side of \((2.9)\) converges. Now,

\[
\sum_{n=p+1}^{\infty} n \{n(1+\beta) - [p(1-\beta) + 2\alpha \beta]\} \frac{(a+1)n-b+11n-p-1}{(c+1)n-p-1(1)n-p}
= \sum_{n=0}^{\infty} (n+p+1) \{(n+p+1)(1+\beta) - [p(1-\beta) + 2\alpha \beta]\} \frac{(a+1)n+b+1}{(c+1)n(1)n+1}
= (1+\beta) \sum_{n=0}^{\infty} (n+1) \frac{(a+1)n+b+1}{(c+1)n(1)n+1} + [p(1+3\beta) - 2\alpha \beta] \sum_{n=0}^{\infty} \frac{(a+1)n+b+1}{(c+1)n(1)n}
+ 2\beta p(p-\alpha) \sum_{n=0}^{\infty} \frac{(a+1)n+b+1}{(c+1)n(1)n+1}
= (1+\beta) \frac{(a+1)(b+1)}{(c+1)} \sum_{n=0}^{\infty} \frac{(a+2)n(b+2)n}{(c+2)n(1)n} + [p(1+3\beta) + (1+\beta) - 2\alpha b] \sum_{n=0}^{\infty} \frac{(a+1)n+b+1}{(c+1)n(1)n+1} + 2\beta p(p-\alpha) \frac{c}{ab} \sum_{n=1}^{\infty} \frac{(a)n(b)n}{(c)n(1)n}
= \frac{\Gamma(c+1)\Gamma(c-a-b-2)}{\Gamma(c-a)\Gamma(c-b)} \left\{ (a+1)(b+1)(1+\beta) + [p(1+3\beta) + (1+\beta) - 2\alpha \beta](c-a-b-2) \right\}
\]

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This last expression is bounded above by \( \| c^{ab} \|^{2} \beta p(p - \alpha) \) if and only if

\[
(a + 1)(b + 1)(1 + \beta) + [p(1 + 3\beta) + (1 + \beta) - 2\alpha \beta](c - a - b - 2)
+ \frac{2\beta p(p - \alpha)}{ab} (c - a - b - 2)^2 \leq 0,
\]

which is equivalent to (2.8).

Putting \( p = \beta = 1 \) in Theorem 4, we obtain the following corollary.

**Corollary 1.** If \( a, b > -1, \ ab < 0, \) and \( c > a + b + 2, \) then \( z^p(a,b;c;z) \) is in the class \( C(\alpha) (0 \leq \alpha < 1) \), if and only if

\[
(a + 1)(b + 1)(1 + \beta) + [p(1 + 3\beta) + (1 + \beta) - 2\alpha \beta](c - a - b - 2)
+ \frac{2\beta p(p - \alpha)}{ab} (c - a - b - 2)^2 \geq 0.
\]

**Remark 1.** Corollary 1 corrects the result given by Silverman [10, Theorem 4].

### 3. An Integral Operator

In this section, we obtain results in connection with a particular integral operator \( G_p(a,b;c;z) \) acting on \( F(a,b;c;z) \) as follows:

\[
G_p(a,b;c;z) = p \int_0^z t^{p-1} F(a,b;c;t) dt
= z^p + \sum_{n=1}^{\infty} \left( \frac{p}{n + p} \right) \frac{(a)_n (b)_n}{(c)_n (1)_n} z^{n+p}.
\]  

(3.1)

We note that \( \frac{zG_p'}{p} = h_p \).

To prove Theorem 5, we shall need the following lemma.

**Lemma 3 [3].** (i) If \( a, b > 0 \) and \( c > a+b \), then a sufficient condition for \( G_p(a,b;c;z) \) defined by (3.1) to be in the class \( S_p^* \) is that

\[
\frac{\Gamma(c)\Gamma(c - a - b)}{\Gamma(c - a)\Gamma(c - b)} \leq 2.
\]

(ii) If \( a, b > -1, c > 0, \) and \( ab < 0, \) then \( G_p(a,b;c;z) \) defined by (3.1) is in the class \( T(p) \) or in the class \( S(p) \) if and only if \( c > \text{max}\{a,b\} \).
Now \( G_p(a, b; c; z) \in K_p(\alpha, \beta) \) if and only if \( \frac{z}{p} G'_p(a, b; c; z) = h_p(a, b; c; z) \in S'_p(\alpha, \beta) \). This follows upon observing that \( \frac{z}{p} G'_p = h_p, \frac{z}{p} G''_p = h'_p - \frac{1}{p} G'_p \), and so

\[
1 + \frac{zG''_p}{G'_p} = \frac{zh'_p}{h_p}.
\]

Thus any \( p \)-valent starlike about \( h_p \) leads to a \( p \)-valent convex function \( G_p \). Thus from Theorems 1, 2 and Lemma 3, we obtain the following theorem.

**Theorem 5.** (i) If \( a, b > 0 \) and \( c > a + b + 1 \), then a sufficient condition for \( G_p(a, b; c; z) \) defined in (3.1) to be in the class \( K_p(\alpha, \beta)(0 \leq \alpha < p, 0 < \beta \leq 1) \) is that

\[
\frac{\Gamma(c)\Gamma(c - a - b)}{\Gamma(c - a)\Gamma(c - b)} \left[ 1 + \frac{ab(1 + \beta)}{2\beta(p - \alpha)(c - a - p - 1)} \right] \leq 2.
\]

(ii) If \( a, b > -1, ab < 0, \) and \( c > a + b + 2 \), then a necessary and sufficient condition for \( G_p(a, b; c; z) \) to be in the class \( C_p(\alpha, \beta) \) is that \( c \geq a + b + 1 - \frac{ab(1 + \beta)}{2\beta(p - \alpha)} \).

**Remark 2.** (i) Putting \( \beta = 1 \) in all the above results we obtain the results, obtained by El-Ashwah et al. [3];

(ii) Putting \( p = \beta = 1 \) in all the above results we obtain the results, obtained by Silverman [10];

(iii) Putting \( p = 1 \) in all the above results we obtain the analogous results, obtained by Mostafa [6].

**References**


Department of Mathematics,
Faculty of Science,
Damietta University
New Damietta 34517, Egypt,
e-mail: r_elashwah@yahoo.com