ON THE EXISTENCE OF POSITIVE SOLUTIONS FOR A CLASS OF SINGULAR NONLINEAR SEMIPOSITONE SYSTEMS

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ABSTRACT. In this paper we consider the existence of positive solutions for singular nonlinear semipositone system of the form

\[
\begin{align*}
-\text{div}(|x|^{-\alpha p}|\nabla u|^{p-2}\nabla u) &= |x|^{-(\alpha+1)p+c_1}(a_1 u^{p-1} - f_1(v) - \frac{b_1}{v^{\gamma_1}}), & x \in \Omega, \\
-\text{div}(|x|^{-\beta q}|\nabla v|^{q-2}\nabla v) &= |x|^{-(\beta+1)q+c_2}(a_2 v^{q-1} - f_2(u) - \frac{b_2}{v^{\gamma_2}}), & x \in \Omega, \\
u = v &= 0, & x \in \partial \Omega,
\end{align*}
\]

where \(\Omega\) is a bounded domain of \(\mathbb{R}^N\) with \(0 \in \Omega\), \(1 < p, q < N\), \(0 \leq \alpha < \frac{N-p}{p}\), \(0 \leq \beta < \frac{N-q}{q}\), \(\gamma_1, \gamma_2 \in (0, 1)\), and \(a_1, a_2, b_1, b_2, c_1, c_2\) are positive parameters. Here \(f_i : [0, \infty) \to \mathbb{R}\) are \(C^2\) functions for \(i = 1, 2\). We discuss the existence of positive solution when \(f_1, f_2\) satisfy certain additional conditions. We use the method of sub-super solutions to establish our results.

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1. INTRODUCTION

We study the existence of positive solutions to the singular infinite semipositone system

\[
\begin{align*}
-\text{div}(|x|^{-\alpha p}|\nabla u|^{p-2}\nabla u) &= |x|^{-(\alpha+1)p+c_1}(a_1 u^{p-1} - f_1(v) - \frac{b_1}{v^{\gamma_1}}), & x \in \Omega, \\
-\text{div}(|x|^{-\beta q}|\nabla v|^{q-2}\nabla v) &= |x|^{-(\beta+1)q+c_2}(a_2 v^{q-1} - f_2(u) - \frac{b_2}{v^{\gamma_2}}), & x \in \Omega, \\
u = v &= 0, & x \in \partial \Omega,
\end{align*}
\]
where $\Omega$ is a bounded smooth domain of $\mathbb{R}^N$ with $0 \in \Omega$, $1 < p, q < N$, $0 \leq \alpha < \frac{N-p}{p}$, $0 \leq \beta < \frac{N-q}{q}$, $\gamma_1, \gamma_2 \in (0, 1)$, and $a_1, a_2, b_1, b_2, c_1, c_2$ positive parameters. Here $f_i : [0, \infty) \to \mathbb{R}$ are continuous functions for $i = 1, 2$. We make the following assumptions:

(A1) There exist $L > 0$ and $b > 0$ such that $f_i(u) < Lu^b$, for all $u \geq 0$ and $i = 1, 2$.

(A2) There exists a constant $S > 0$ such that $\max\{a_1u^{p-1} - f_1(v), a_2v^{q-1} - f_2(u)\} < S$ for all $u, v \geq 0$.

Elliptic problems involving more general operator, such as the degenerate quasilinear elliptic operator given by $-\text{div}(|x|^{-\alpha p}\nabla u|^{p-2}\nabla u)$, were motivated by the following Caffarelli, Kohn and Nirenberg’s inequality (see [1], [2]). The study of this type of problem is motivated by its various applications, for example, in fluid mechanics, in newtonian fluids, in flow through porous media and in glaciology (see [3], [4]). So, the study of positive solutions of singular elliptic problems has more practical meanings. We refer to [5], [6], [7], [8] for additional results on elliptic problems. Here we focus on further extending the single equation in [9] to the system (1). Our approach is based on the method of sub-super solutions, see [10, 11].

2. Preliminaries and existing result

In this paper, we denote $W^{1,p}_0(\Omega, |x|^{-\alpha p})$, the completion of $C_0^\infty(\Omega)$, with respect to the norm $\|u\| = \left(\int_\Omega |x|^{-\alpha p}|\nabla u|^p dx\right)^\frac{1}{p}$. To precisely state our existence result we consider the eigenvalue problem

$$\begin{cases} 
-\text{div}(|x|^{-sp}\nabla \phi|^{r-2}\nabla \phi) = \lambda |x|^{-\alpha p}|x|^{-t}\phi|^{p-2}\phi, & x \in \Omega, \\
\phi = 0, & x \in \partial \Omega.
\end{cases} \quad (2)$$

For $r = p$, $s = \alpha$ and $t = c_1$, let $\phi_{1,p}$ be the eigenfunction corresponding to the first eigenvalue $\lambda_{1,p}$ of (2) such that $\phi_{1,p}(x) > 0$ in $\Omega$ and $\|\phi_{1,p}\| = 1$ and for $r = q$, $s = \beta$ and $t = c_2$, let $\phi_{1,q}$ be the eigenfunction corresponding to the first eigenvalue $\lambda_{1,q}$ of (2) such that $\phi_{1,q}(x) > 0$ in $\Omega$, and $\|\phi_{1,q}\| = 1$ (see [12, 13]). It can be shown that $\frac{\partial \phi_{1,r}}{\partial n} < 0$ on $\partial \Omega$ for $r = p, q$. Here $n$ is the outward normal. We will also consider the unique solution $(\zeta_p(x), \zeta_q(x)) \in W_0^0(\Omega, |x|^{-\alpha p}) \times W_0^0(\Omega, |x|^{-\beta q})$ for the system

$$\begin{cases} 
-\text{div}(|x|^{-\alpha p}\nabla \zeta_p|^{p-2}\nabla \zeta_p) = |x|^{-\alpha p}|x|^{-t}\zeta_p|^{p-2}\zeta_p, & x \in \Omega, \\
-\text{div}(|x|^{-\beta q}\nabla \zeta_q|^{q-2}\nabla \zeta_q) = |x|^{-\beta q}|x|^{-t}\zeta_q|^{q-2}\zeta_q, & x \in \Omega, \\
\zeta_p = \zeta_q = 0, & x \in \partial \Omega,
\end{cases}$$

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to discuss our existence result. It is well known that \( \zeta_r(x) > 0 \) in \( \Omega \) and \( \frac{\partial \zeta_r(x)}{\partial n} < 0 \) on \( \partial \Omega \), for \( r = p, q \) (see [12]).

A pair of nonnegative functions \( (\psi_1, \psi_2), (z_1, z_2) \) are called a sub-solution and super-solution of (1) if they satisfy \( (\psi_1, \psi_2) = (0, 0) = (z_1, z_2) \) on \( \partial \Omega \) and

\[
\int_{\Omega} |x|^{-\alpha p} |\nabla \psi_1|^p - 2 \nabla \psi_1 \cdot \nabla w \, dx \leq \int_{\Omega} |x|^{-(\alpha+1)p+c_1}(a_1 \psi_1^{p-1} - f_1(\psi_2) - \frac{b_1}{\psi_1^q}) \, w \, dx,
\]

\[
\int_{\Omega} |x|^{-\beta q} |\nabla \psi_2|^q - 2 \nabla \psi_2 \cdot \nabla w \, dx \leq \int_{\Omega} |x|^{-(\beta+1)q+c_2}(a_2 \psi_2^{q-1} - f_2(\psi_1) - \frac{b_2}{\psi_2^p}) \, w \, dx,
\]

\[
\int_{\Omega} |x|^{-\alpha p} |\nabla z_1|^p - 2 \nabla z_1 \cdot \nabla w \, dx \geq \int_{\Omega} |x|^{-(\alpha+1)p+c_1}(a_1 z_1^{p-1} - f_1(z_2) - \frac{b_1}{z_1^q}) \, w \, dx,
\]

\[
\int_{\Omega} |x|^{-\beta q} |\nabla z_2|^q - 2 \nabla z_2 \cdot \nabla w \, dx \geq \int_{\Omega} |x|^{-(\beta+1)q+c_2}(a_2 z_2^{q-1} - f_2(z_1) - \frac{b_2}{z_2^p}) \, w \, dx.
\]

for all \( w \in W = \{ w \in C_0^\infty(\Omega) \mid w \geq 0, x \in \Omega \} \). Then the following result holds:

**Lemma 1.** (see [12]). Suppose there exist sub and super-solutions \( (\psi_1, \psi_2) \) and \( (z_1, z_2) \) respectively of (1) such that \( (\psi_1, \psi_2) \leq (z_1, z_2) \). Then (1) has a solution \((u, v)\) such that \((u, v) \in [(\psi_1, \psi_2), (z_1, z_2)]\).

**Theorem 2.** Assume if \( a_1 > \left( \frac{p}{p-1+\gamma_1} \right)^{p-1} \lambda_{1,p}, \ a_2 > \left( \frac{q}{q-1+\gamma_2} \right)^{q-1} \lambda_{1,q} \), then there exists \( c > 0 \) such that if \( \max\{b_1, b_2\} \leq c \), then the system (1) admits a positive solution.

**Proof.** We start with the construction of a positive subsolution for (1). To get a positive subsolution, we can apply an anti-maximum principle (see [14]), from which we know that there exist a \( \delta_1 > 0 \) and a solution \( z_\lambda \) of

\[
\begin{cases}
-\text{div}(|x|^{-s} |\nabla z|^r - 2 \nabla z) = |x|^{-(s+1)r+t}(\lambda z^{r-1} - 1), & x \in \Omega, \\
z = 0, & x \in \partial \Omega,
\end{cases}
\]

for \( \lambda \in (\lambda_{1,r}, \lambda_{1,r} + \delta_1) \), for \( r = p, q, s = \alpha, \beta \) and \( t = c_1, c_2 \).

Fix \( \lambda_1 \in \left( \lambda_{1,p}, \min\left\{ \left( \frac{p-1+\gamma_1}{p} \right)^{p-1} a_1, \lambda_{1,p} + \delta_1 \right\} \right) \) and

\( \dot{\lambda}_2 \in \left( \lambda_{1,q}, \min\left\{ \left( \frac{q-1+\gamma_2}{q} \right)^{q-1} a_2, \lambda_{1,q} + \delta_1 \right\} \right) \). Let \( \theta_i = \|z_{\lambda_i}\| \) for \( i = 1, 2 \). It is well known that \( z_{\lambda_1}, z_{\lambda_2} > 0 \) in \( \Omega \) and \( \frac{\partial z_{\lambda_1}}{\partial n}, \frac{\partial z_{\lambda_2}}{\partial n} < 0 \) on \( \partial \Omega \), where \( n \) is the outer unit normal to \( \Omega \). Hence there exist positive constants \( \epsilon, \delta, \sigma_p, \sigma_q \) such that

\[
|x|^{-s} |\nabla z_{\lambda_i}|^r \geq \epsilon, \quad x \in \overline{\Omega}_\delta,
\]

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\[ z_{\lambda_1} \geq \sigma_r, \quad x \in \Omega_0 = \Omega \setminus \Omega_\delta, \] (5)

with \( r = p, q; \ s = \alpha, \beta; \ i = 1, 2 \) and \( \Omega_\delta = \{x \in \Omega \mid d(x, \partial \Omega) \leq \delta \} \). Choose \( \eta_1, \eta_2 > 0 \) such that \( \eta_1 \leq \min \{ |x|^{-(s+1)p+t}, \eta_2 \geq \max \{ |x|^{-(s+1)p+t}, \in \Omega_\delta \}, \) for \( r = p, q, s = \alpha, \beta \) and \( t = c_1, c_2 \). We construct a subsolution \((\psi_1, \psi_2)\) of (1) using \( z_{\lambda_1}, z_{\lambda_2} \).

Define \((\psi_1, \psi_2) = \left( M\left( \frac{p-1+\gamma_1}{p}\right) z_{\lambda_1}^{\frac{1-\gamma_1}{p}}, M\left( \frac{q-1+\gamma_2}{q}\right) z_{\lambda_2}^{\frac{1-\gamma_2}{q}} \right), \)
where

\[
M = \min \left\{ \left( \frac{p-1+\gamma_1}{p}\right)^{\gamma_1} \left( \frac{q-1+\gamma_2}{q}\right)^{\gamma_2} \lambda_1, \left( \frac{p-1+\gamma_1}{p}\right)^{\gamma_1} \left( \frac{q-1+\gamma_2}{q}\right)^{\gamma_2} \lambda_2 \right\}. 
\]

Let \( w \in W \). Then a calculation shows that

\[
\nabla \psi_1 = M z_{\lambda_1}^{\frac{1-\gamma_1}{p}} \nabla z_{\lambda_1},
\]

\[
\int_{\Omega} |x|^{-\alpha p} |\nabla \psi_1|^{p-2} \nabla \psi_1 \cdot \nabla w dx = M^{p-1} \int_{\Omega} |x|^{-\alpha p} z_{\lambda_1}^{\frac{1-\gamma_1}{p}} |\nabla z_{\lambda_1}|^{p-2} \nabla z_{\lambda_1} \nabla w dx
\]

\[
= M^{p-1} \int_{\Omega} |x|^{-\alpha p} |\nabla z_{\lambda_1}|^{p-2} \nabla z_{\lambda_1} \left[ \nabla \left( z_{\lambda_1}^{\frac{1-\gamma_1}{p}} w \right) - \left( \nabla z_{\lambda_1}^{\frac{1-\gamma_1}{p}} \right) w \right] dx
\]

\[
= M^{p-1} \int_{\Omega} |x|^{-\alpha (1) p+c_1} M^{p-1} \left( \frac{1-\gamma_1}{p} \right) \lambda_1 z_{\lambda_1}^{p-1} - |x|^{-\alpha p} \left( \frac{1-\gamma_1}{p} \right) \| \nabla z_{\lambda_1} \|^{p} \frac{1-\gamma_1}{p} \lambda_1 z_{\lambda_1}^{p-1} \right) w dx
\]

\[
= \int_{\Omega} |x|^{-\alpha (1) p+c_1} M^{p-1} \lambda_1 z_{\lambda_1}^{p-1} \left[ a_1 \psi^{p-1} - f_1(\psi_2) - \frac{b_1}{\psi_1^p} \right] w dx,
\]

and

\[
\int_{\Omega} |x|^{-\alpha (1) p+c_1} a_1 \psi^{p-1} - f_1(\psi_2) - \frac{b_1}{\psi_1^p} w dx =
\]

\[
\int_{\Omega} |x|^{-\alpha (1) p+c_1} a_1 M^{p-1} \left( \frac{p-1+\gamma_1}{p} \right) z_{\lambda_1}^{p-1} - |x|^{-\alpha (1) p+c_1} f_1 \left( M\left( \frac{q-1+\gamma_2}{q} \right) z_{\lambda_2}^{\frac{1-\gamma_2}{q}} \right)
\]

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\[-|x|^{-(\alpha+1)p+c_1} \frac{b_1}{M^{\gamma_1} \left( \frac{p-1+\gamma_1}{p} \right)^{\gamma_1} z^{\alpha+1} \gamma_1}} \int \Omega w dx. \] (7)

Similarly
\[\int_{\Omega} |x|^{-\beta q} |\nabla \psi_2|^q dx = \int_{\Omega} |x|^{-(\beta+1)q+c_2} M^{q-1} \frac{\gamma_1}{\gamma_2} w dx, \] (8)

and
\[\int_{\Omega} |x|^{-(\beta+1)q+c_2} \left[ a_2 \psi_2^{\beta-1} - f_2(\psi_2) \right] dx = \int_{\Omega} |x|^{-(\beta+1)q+c_2} M^{q-1} \left( \frac{q-1+\gamma_2}{q} \right)^{\gamma_1} w dx, \] (9)

Let \( c = \min \left\{ M^{\beta-1+\gamma_1} \left( \frac{1-\gamma_1}{p-1+\gamma_1} \right)^{\gamma_1} \right\}, \) then we have
\[|x|^{-(\alpha+1)p+c_1} M^{\beta-1+\gamma_1} z^{\alpha+1} \gamma_1 \leq |x|^{-(\alpha+1)p+c_1} a_1 M^{p-1} \left( \frac{p-1+\gamma_1}{p} \right)^{p-1} z^{\alpha+1} \gamma_1, \] (10)

and from the choice of \( M, \) we know that
\[LM^{b-p+1} \theta_2 \frac{\phi}{q-1+\gamma_2} \leq \left( \frac{q}{q-1+\gamma_2} \right)^{p-1} \frac{2^{(1-\gamma_1)(p-1)}}{p^{1-\gamma_1}}. \] (11)

By (11) and (A1) we have
\[-|x|^{-(\alpha+1)p+c_1} M^{\beta-1} \left( \frac{1-\gamma_1}{p-1+\gamma_1} \right)^{p-1} \leq -|x|^{-(\alpha+1)p+c_1} \left| \nabla \psi_2 \right| \leq -|x|^{-(\alpha+1)p+c_1} M^{b} \left( \frac{q-1+\gamma_2}{q} \right)^{p-1} \frac{\phi}{q-1+\gamma_2}, \] (12)

Next, from (7) and definition of \( c, \) we have
\[|x|^{-\alpha p} M^{p-1} \left( \frac{1-\gamma_1}{p-1+\gamma_1} \right)^{\gamma_1} \leq |x|^{-\alpha p+1} c_1 \frac{b_1}{M^{\gamma_1} \left( \frac{p-1+\gamma_1}{p} \right)^{\gamma_1}}, \]
and

\[- |x|^{-\alpha} p M^{p-1} \frac{(1 - \gamma_1)(p - 1)}{p - 1 + \gamma_1} \frac{|\nabla z_{x_1}|^p}{z_{x_1}^{\frac{p(1 - \gamma_1)}{p - 1 + \gamma_1}}} \leq - |x|^{-(\alpha + 1) p + c_1} \frac{b_1}{M^{\gamma_1} \frac{(p - 1 + \gamma_1)}{p} \frac{p - 1}{p - 1 + \gamma_1} \frac{1}{z_{x_1}^{\frac{p(1 - \gamma_1)}{p - 1 + \gamma_1}}}}.
\]

Hence by using (10), (12) and (13) for \( b_1 \leq c \), we have

\[
\int_{\Omega_0} |x|^{-\alpha} p |\nabla \psi_1|^2 \nabla \psi_1 \cdot \nabla \psi \, dx \leq \int_{\Omega_0} \left[ |x|^{-(\alpha + 1) p + c_1} a_1 M^{p-1} \left( \frac{p - 1 + \gamma_1}{p} \frac{p - 1}{p} \frac{1}{z_{x_1}^{\frac{p(1 - \gamma_1)}{p - 1 + \gamma_1}}} - |x|^{-(\alpha + 1) p + c_1} f_1 \left( M \left( \frac{q - 1 + \gamma_2}{q} \frac{z_{x_1}^{\frac{p(1 - \gamma_1)}{p - 1 + \gamma_1}}} \right) - |x|^{-(\alpha + 1) p + c_1} \frac{b_1}{M^{\gamma_1} \frac{(p - 1 + \gamma_1)}{p} \frac{p - 1}{p} \frac{1}{z_{x_1}^{\frac{p(1 - \gamma_1)}{p - 1 + \gamma_1}}}} \right) \right] \, dx
\]

\[
= \int_{\Omega_0} |x|^{-(\alpha + 1) p + c_1} \left[ a_1 \psi_1^{p-1} - f_1 (\psi_2) - \frac{b_1}{\psi_1^2} \right] \, dx.
\]

Similarly

\[
\int_{\Omega_0} |x|^{-\alpha} p |\nabla \psi_2|^2 \nabla \psi_2 \cdot \nabla \psi \, dx \leq \int_{\Omega_0} \left[ |x|^{-(\alpha + 1) p + c_2} \frac{a_2 M^{q-1} \left( \frac{q - 1 + \gamma_2}{q} \frac{z_{x_1}^{\frac{p(1 - \gamma_1)}{p - 1 + \gamma_1}}} \right) - |x|^{-(\alpha + 1) p + c_2} \frac{b_2}{M^{\gamma_2} \frac{(q - 1 + \gamma_2)}{q} \frac{z_{x_1}^{\frac{p(1 - \gamma_1)}{p - 1 + \gamma_1}}} \right] \, dx
\]

\[
= \int_{\Omega_0} |x|^{-(\alpha + 1) p + c_2} \left[ a_2 \psi_2^{q-1} - f_2 (\psi_1) - \frac{b_2}{\psi_2^2} \right] \, dx.
\]

On the other hand, on \( \Omega_0 = \Omega \setminus \Omega_\delta \), we have \( z_{x_1} \geq \sigma_p \) and \( z_{x_2} \geq \sigma_q \), for some \( 0 < \sigma_p, \sigma_q < 1 \), and from the definition of \( c \), for \( b_1 \leq c \) we have

\[
\frac{b_1}{M^{\gamma_1} \frac{(p - 1 + \gamma_1)}{p} \frac{p - 1}{p} \frac{1}{z_{x_1}^{\frac{p(1 - \gamma_1)}{p - 1 + \gamma_1}}}} \leq \frac{1}{p} M^{p-1} p \left[ (\frac{p - 1 + \gamma_1}{p})^{p - 1} a_1 - \lambda_1 \right] \leq \frac{1}{p} M^{p-1} p \left[ (\frac{p - 1 + \gamma_1}{p})^{p - 1} a_1 - \lambda_1 \right].
\]

Also from the choice of \( M \), we have

\[
LM^{b-1} \left( \frac{q - 1 + \gamma_2}{q} \right) z_{x_1}^{\frac{q(p-1)}{q - 1 + \gamma_2}} \leq \frac{b}{z_{x_1}^{\frac{1}{\gamma_1}} \frac{p - 1}{p - 1 + \gamma_1} \frac{1}{z_{x_1}^{\frac{p(1 - \gamma_1)}{p - 1 + \gamma_1}}}} \leq \frac{1}{p} M^{p-1} p \left[ (\frac{p - 1 + \gamma_1}{p})^{p - 1} a_1 - \lambda_1 \right].
\]

Hence from (16) and (17) we have

\[
\int_{\Omega_0} |x|^{-\alpha} p |\nabla \psi_1|^2 \nabla \psi_1 \cdot \nabla \psi \, dx = \int_{\Omega_0} \left[ |x|^{-(\alpha + 1) p + c_1} \frac{a_1 M^{p-1} \left( \frac{p(1 - \gamma_1)}{p - 1 + \gamma_1} \right)}{z_{x_1}^{\frac{p(1 - \gamma_1)}{p - 1 + \gamma_1}}} - |x|^{-(\alpha + 1) p + c_1} \frac{b_1}{z_{x_1}^{\frac{p(1 - \gamma_1)}{p - 1 + \gamma_1}}} \right] \, dx
\]

\[
\leq \int_{\Omega_0} |x|^{-(\alpha + 1) p + c_1} \frac{a_1 M^{p-1} \left( \frac{p(1 - \gamma_1)}{p - 1 + \gamma_1} \right)}{z_{x_1}^{\frac{p(1 - \gamma_1)}{p - 1 + \gamma_1}}} \, dx = \int_{\Omega_0} |x|^{-(\alpha + 1) p + c_1} \frac{1}{z_{x_1}^{\frac{p(1 - \gamma_1)}{p - 1 + \gamma_1}}} \left[ \frac{1}{p} L M^{p-1} \frac{p - 1}{p - 1 + \gamma_1} \right] \, dx
\]
Similarly, we shall verify that \( (\psi, z) \) in \( \Omega \). Hence, if \( \max \{ b_1, b_2 \} \leq c \), by Lemma 1 there exists a positive solution \((u, v)\) of (1) such that \((\psi_1, \psi_2) \leq (u, v) \leq (z_1, z_2)\). This completes the proof of Theorem 2.

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