CERTAIN SUBCLASS OF BESSEL FUNCTIONS WITH RESPECT TO \((j,k)\)-SYMMETRIC POINTS

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Abstract. In this paper, the authors introduce new class of analytic functions with respect to \((j,k)\)-symmetric points and investigate various inclusion properties for these classes. Integral representation for functions in these classes and some interesting applications involving a familiar integral operator, are also obtained.

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1. Introduction

Observations: Let \(\mathcal{A}\) denote the class of functions of the form

\[
f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad a_n \geq 0,
\]

which are analytic in the open disc \(\mathcal{U} = \{ z : z \in \mathbb{C} : |z| < 1 \}\) and \(\mathcal{S}\) be the class of functions \(f \in \mathcal{A}\) which are univalent in \(\mathcal{U}\).

We denote by \(\mathcal{S}^*, \mathcal{C}, \mathcal{K}\) and \(\mathcal{C}^*\) the familiar subclasses of \(\mathcal{A}\) consisting of functions which are respectively starlike, convex, close-to-convex and quasi-convex in \(\mathcal{U}\). Our favorite references of the field are ([3], [6]) which covers most of the topics in a lucid and economical style.

In [9], Rønning introduced a new class of starlike functions related to \(UCV\) defined as

\[
f(z) \in \mathcal{S}_P \iff \text{Re}\left( \frac{zf'(z)}{f(z)} \right) > \left| \frac{zf'(z)}{f(z)} - 1 \right|.
\]

Note that \(f(z) \in UCV \iff zf'(z) \in \mathcal{S}_P\). The geometrical interpretation of uniformly convex and related functions have been studied by several authors (see [4, 5, 9]).
An analytic function \( f \) is said to be subordinate to an analytic function \( g \) (written as \( f \prec g \)) if and only if there exists an analytic function \( \omega \) with
\[
\omega(0) = 0 \text{ and } |\omega(z)| < 1 \text{ for } z \in \mathcal{U},
\]
such that
\[
f(z) = g(\omega(z)) \text{ for } z \in \mathcal{U}.
\]
In particular, if \( g \) is univalent in \( \mathcal{U} \), we have the following equivalence
\[
f \prec g \iff f(0) = g(0) \text{ and } f(\mathcal{U}) \subset g(\mathcal{U}).
\]
The convolution or Hadamard product of two functions of class \( \mathcal{A} \) is denoted and defined by
\[
(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n,
\]
where \( f \) has the form (1) and
\[
g(z) = z + \sum_{n=2}^{\infty} b_n z^n \quad z \in \mathcal{U}.
\]
Let us consider the following second-order linear homogeneous differential equation (see for details [1] and [2]):
\[
z^2 \omega''(z) + b z \omega'(z) + \left[ d z^2 - v^2 + (1 - b) v \right] \omega(z) = 0 \quad (v, b, d \in \mathbb{C}). \tag{2}
\]
The function \( \omega_{v,b,d}(z) \), which is called the generalized Bessel function of the first kind of order \( v \), is defined as a particular solution of (2). The function \( \omega_{v,b,d}(z) \) has the familiar representation as
\[
\omega_{v,b,d}(z) = \sum_{n=0}^{\infty} \frac{(-d)^n}{n! \Gamma(v + n + \frac{b+1}{2})} \left( \frac{z}{2} \right)^{2n+v} \quad (z \in \mathbb{C}). \tag{3}
\]
Here \( \Gamma \) stands for the Euler gamma function. The series (3) permits the study of Bessel, modified Bessel, and spherical Bessel function altogether. It is worth mentioning that, in particular:

(i) For \( b = d = 1 \) in (3), we obtain the familiar Bessel function of the first kind of order \( U \) defined by
\[
\mathcal{J}_v(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n! \Gamma(v + n + 1)} \left( \frac{z}{2} \right)^{2n+v} \quad (z \in \mathbb{C}). \tag{4}
\]
(ii) For $b = 1$ and $d = -1$ in (3), we obtain the modified Bessel function of the first kind of order $v$ defined by

$$I_v(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n! \Gamma(v + n + 1)} \left( \frac{z}{2} \right)^{2n+v} \quad (z \in \mathbb{C}). \quad (5)$$

(iii) For $b = 2$ and $d = 1$ in (3), the function $\omega_{v,b,d}(z)$ reduces to $\sqrt{\frac{2}{\pi}} S_v(z)$, where $S_v$ is the spherical Bessel function of the first kind of order $v$, defined by

$$S_v(z) = \frac{\sqrt{\pi}}{\sqrt{2}} \sum_{n=0}^{\infty} \frac{(-1)^n}{n! \Gamma(v + n + \frac{3}{2})} \left( \frac{z}{2} \right)^{2n+v} \quad (z \in \mathbb{C}). \quad (6)$$

Now, consider the function $u_{v,b,d}(z) : \mathbb{C} \rightarrow \mathbb{C}$, defined by the transformation

$$u_{v,b,d}(z) = 2^v \Gamma \left( v + \frac{b+1}{2} \right) z^{-\frac{v}{2}} \omega_{v,b,d}(\sqrt{z}). \quad (7)$$

By using the well-known Pochhammer symbol (or the shifted factorial) $(\lambda)_\mu$ defined, for $\lambda, \mu \in \mathbb{C}$ and in terms of the Euler gamma function, by

$$(\lambda)_\mu := \frac{\Gamma(\lambda + \mu)}{\Gamma(\lambda)} = \begin{cases} 1 & (\mu = 0; \lambda \in \mathbb{C} \setminus \{0\}), \\ \lambda(\lambda + 1) \cdots (\lambda + n - 1) & (\mu = n \in \mathbb{N}; \lambda \in \mathbb{C}), \end{cases}$$

and $(\lambda)_0 = 1$, we obtain for the function $u_{v,b,d}(z)$ the following representation

$$u_{v,b,d}(z) = \sum_{n \geq 0} \frac{(-d)^n}{(v + \frac{b+1}{2})_n} \frac{z^n}{n!},$$

where $k = v + \frac{b+1}{2} \neq 0, -1, -2, \ldots$. This function is analytic on $\mathbb{C}$ and satisfies the second order linear differential equation

$$4z^2 u''(z) + 2(2v + b + 1)zu'(z) + dzu(z) = 0.$$ 

Now, we introduce the function $\varphi_{v,b,d}(z)$ defined in terms of generalized Bessel function $\omega_{v,b,d}(z)$, by

$$\varphi_{v,b,d}(z) = z u_{v,b,d}(z)$$

$$= 2^v \Gamma \left( v + \frac{b+1}{2} \right) z^{1-\frac{v}{2}} \omega_{v,b,d}(\sqrt{z})$$

$$= z + \sum_{n=1}^{\infty} \frac{(-d)^n z^{n+1}}{4^n n! (c)_n}, \ \text{where} \ c = \left( v + \frac{b+1}{2} \right)$$

$$= g(c, d, z).$$
Motivated by Selvaraj and Karthikeyan [11], we define the following $D_m^\lambda(c,d) f(z) : \mathcal{U} \rightarrow \mathcal{U}$ by

$$D_\lambda(c,d) f(z) = f(z) \ast g(c,d,z)$$

$$D_1^\lambda(c,d) f(z) = (1 - \lambda) (f(z) \ast g(c,d,z)) + \lambda z (f(z) \ast g(c,d,z))'$$

$$D_m^\lambda(c,d) f(z) = D_1^\lambda (D_{m-1}^\lambda(c,d) f(z))$$

If $f \in \mathcal{A}$, then from (9) and (10) we may easily deduce that

$$D_m^\lambda(c,d) f(z) = z + \sum_{n=2}^{\infty} \frac{(1 + (n - 1)\lambda)^m (-d)^{n-1} a_n z^n}{4^{n-1}(n-1)! (c)_n},$$

where $m \in N_0 = N \cup \{0\}$ and $\lambda \geq 0$.

It can be easily verified from definition of (11) that

$$\lambda z (D_m^\lambda(c,d) f(z))' = D_{m+1}^\lambda(c,d) f(z) - (1 - \lambda) D_m^\lambda(c,d) f(z).$$

Let $k$ be a positive integer and $j = 0, 1, 2, \ldots (k - 1)$. A domain $D$ is said to be $(j,k)$–fold symmetric if a rotation of $D$ about the origin through an angle $2\pi j/k$ carries $D$ onto itself. A function $f \in \mathcal{A}$ is said to be $(j,k)$–symmetrical if for each $z \in \mathcal{U}$

$$f(\varepsilon z) = \varepsilon^j f(z),$$

where $\varepsilon = \exp(2\pi i/k)$. The family of $(j,k)$–symmetrical functions will be denoted by $\mathcal{F}_{j,k}^1$. We observe that $\mathcal{F}_{1,2}^1, \mathcal{F}_{2,2}^1$ and $\mathcal{F}_{1,k}^1$ are well-known families of odd functions, even functions and $k$-symmetrical functions respectively.

Also let $f_{j,k}(z)$ be defined by the following equality

$$f_{j,k}(z) = \frac{1}{k} \sum_{v=0}^{k-1} \frac{f(\varepsilon^v z)}{\varepsilon^{vj}}, \quad (f \in \mathcal{A}; k = 1, 2, \ldots; j = 0, 1, 2, \ldots (k - 1)),

where $v$ is an integer.

The notation of $(j,k)$–symmetrical functions was introduced and studied by Liczberski and Polubinski in [8].

The following identities follow directly from (14):

$$f_{j,k}(\varepsilon^v z) = \varepsilon^{vj} f_{j,k}(z),$$

$$f'_{j,k}(\varepsilon^v z) = \varepsilon^{vj-v} f'_{j,k}(z) = \frac{1}{k} \sum_{v=0}^{k-1} \frac{f'(\varepsilon^v z)}{\varepsilon^{vj-v}},$$

$$f''_{j,k}(\varepsilon^v z) = \varepsilon^{vj-2v} f''_{j,k}(z) = \frac{1}{k} \sum_{v=0}^{k-1} \frac{f''(\varepsilon^v z)}{\varepsilon^{vj-2v}}.$$
Motivated by the concept introduced by K. Sakaguchi in [10], recently several subclasses of analytic functions with respect to $k$-symmetric points were introduced and studied by various authors. In this paper, new subclasses of analytic functions with respect to $(j,k)$-symmetric points are introduced.

Now we define

$$f_{j,k}^{\lambda,m}(c,d;z) = \frac{1}{k} \sum_{v=0}^{k-1} e^{-\nu_j} (D_{\lambda}^{m}(c,d)f(\varepsilon^v z)), \quad (f \in A; k = 1, 2, \ldots; j = 0, 1, 2, \ldots (k-1)).$$

(16)

Clearly for $j = k = 1$, we have

$$f_{j,k}^{\lambda,m}(c,d;z) = D_{\lambda}^{m}(c,d)f(\varepsilon^v z).$$

**Definition 1.** The class $S_{\lambda,m}^{\lambda,m}(c,d,\alpha,\beta,\gamma)$ of function $f$, analytic in $U$ given by (1) and satisfying the condition

$$\text{Re} \left\{ e^{i\alpha} \left( 1 + \frac{1}{\gamma} \left( \frac{z[D_{\lambda}^{m}(c,d)f(z)]'}{f_{j,k}^{\lambda,m}(c,d,z)} - 1 \right) \right) \right\} > 1 + \frac{1}{\gamma} \left( \frac{z[D_{\lambda}^{m}(c,d)f(z)]'}{f_{j,k}^{\lambda,m}(c,d,z)} - 1 \right), \quad (z \in U),$$

(17)

where $-\pi/2 < \alpha < \pi/2$, $\gamma \in \mathbb{C} \setminus \{0\}$ and $f_{j,k}^{\lambda,m}$ is defined by the equality (16).

**Remark 1.** If we let $j = k = 1$ and $\alpha = \beta = 0$, $\gamma = 1$ in (17), the class $S_{\lambda,m}^{\lambda,m}(c,d,\alpha,\beta,\gamma)$ reduces to the function class $S_p$.

### 2. Integral Representation

**Theorem 1.** If $f \in S_{\lambda,m}^{\lambda,m}(c,d,\alpha,\beta,\gamma)$, then $f_{j,k}^{\lambda,m} \in S^*.$

**Proof.** Let $f \in S_{\lambda,m}^{\lambda,m}(c,d,\alpha,\beta,\gamma)$. For $\omega = u + iv$, the inequality (17) can be rewritten as

$$u > \frac{1}{2} \left( v^2 + 1 - \frac{\beta}{\cos^2 \alpha} \right).$$

Setting

$$G = \left\{ u + iv : u > \frac{1}{2} \left( v^2 + 1 - \frac{\beta}{\cos^2 \alpha} \right) \right\}.$$

From the equivalent subordination condition proved by N. Xu and D. Yang in [15], we may rewrite the conditions (17) in the form

$$1 + \frac{1}{\gamma} \left( \frac{z[D_{\lambda}^{m}(c,d)f(z)]'}{f_{j,k}^{\lambda,m}(c,d,z)} - 1 \right) < e^{-i\alpha} [h(z) \cos \alpha + i \sin \alpha],$$

(18)
where
\[ h(z) = 1 - \frac{\beta}{2 \cos^2 \alpha} + \frac{2}{\pi^2} \left( \log \frac{1 + \sqrt{(z + \theta)(1 + \theta z)}}{1 - \sqrt{(z + \theta)(1 + \theta z)}} \right)^2 \]
with
\[ \theta = \left( \frac{e^\mu - 1}{e^\mu + 1} \right)^2, \mu = \sqrt{\beta \pi/2} \cos \alpha. \]

The function \( e^{-i\alpha} [h(\omega(z)) \cos \alpha + i \sin \alpha] \) is univalent and convex in \( U \), where \( \omega(z) \) is a Schwarz function, analytic in \( U \) with \( \omega(0) = 0 \).

If we replace \( z \) by \( \varepsilon z \) in (18), then (18) will be of the form
\[ \varepsilon^v z \frac{[D_m^\lambda(c, d)f(z)]'}{f_{j,k}^\lambda m(c, d; z)} = \gamma \left( e^{-i\alpha} [h(\omega(\varepsilon z)) \cos \alpha + i \sin \alpha] - 1 \right) + 1. \] (19)

Using (13) in (19), we get
\[ \varepsilon^v \varepsilon^v z \frac{[D_m^\lambda(c, d)f(\varepsilon z)]'}{f_{j,k}^\lambda m(c, d; \varepsilon z)} = \gamma \left( e^{-i\alpha} [h(\omega(\varepsilon^v z)) \cos \alpha + i \sin \alpha] - 1 \right) + 1, \] (20)

Let \( v = 0, 1, 2, \ldots, k - 1 \) in (20) respectively and summing them, we get
\[ \sum_{v=0}^{k-1} \varepsilon^v \varepsilon^v z \frac{[D_m^\lambda(c, d)f(\varepsilon z)]'}{f_{j,k}^\lambda m(c, d; \varepsilon z)} = \sum_{v=0}^{k-1} \gamma \left( e^{-i\alpha} [h(\omega(\varepsilon^v z)) \cos \alpha + i \sin \alpha] - 1 \right) + 1, \]
or equivalently
\[ \frac{z [f_{j,k}^\lambda m(c, d; z)]'}{f_{j,k}^\lambda m(c, d; z)} = \frac{\gamma}{\kappa} \sum_{v=0}^{k-1} \left( e^{-i\alpha} [h(\omega(\varepsilon^v z)) \cos \alpha + i \sin \alpha] - 1 \right) + 1, \]
that is \( f_{j,k}^\lambda m \in S^*. \)
\textbf{Theorem 2.} If \( f \in S_{j,k}^{\lambda,m} (c, d, \alpha, \beta, \gamma) \), then we have

\[
f_{j,k}^{\lambda,m} (c, d; z) = z \exp \left\{ \frac{\gamma}{k} \sum_{v=0}^{k-1} \int_0^{e^v z} \frac{e^{-i\alpha} [h(\omega(\varepsilon^v t)) \cos \alpha + i \sin \alpha] - 1}{t} \, dt \right\},
\]

where \( f_{j,k}^{\lambda,m} (z) \) is defined by (16), \( \omega(z) \) is analytic in \( \mathcal{U} \) and \( \omega(0) = 0, |\omega(z)| < 1 \).

\textit{Proof.} Let \( f \in S_{j,k}^{\lambda,m} (c, d, \alpha, \beta, \gamma) \). In view of (18), we have

\[
z \frac{[D_{\lambda}^{m}(c, d)f(z)']}{f_{j,k}^{\lambda,m}(c, d; z)} = \gamma \left( e^{-i\alpha} [h(\omega(z)) \cos \alpha + i \sin \alpha] - 1 \right) + 1,
\]

(22)

where \( \omega(z) \) is analytic in \( \mathcal{U} \) and \( \omega(0) = 0, |\omega(z)| < 1 \). Substituting \( z \) by \( \varepsilon^v z \) in the equality (22) respectively \((v = 0, 1, 2, \ldots, k - 1, \varepsilon^k = 1)\), we have

\[
e^{v z} \frac{[D_{\lambda}^{m}(c, d)f(\varepsilon^v z)']}{f_{j,k}^{\lambda,m}(c, d; \varepsilon^v z)} = \gamma \left( e^{-i\alpha} [h(\omega(\varepsilon^v z)) \cos \alpha + i \sin \alpha] - 1 \right) + 1.
\]

(23)

Using (13) in (23) can be rewritten in the form

\[
\frac{e^{v - v_j} z[D_{\lambda}^{m}(c, d)f(\varepsilon^v z)']}{f_{j,k}^{\lambda,m}(c, d; \varepsilon^v z)} = \gamma \left( e^{-i\alpha} [h(\omega(\varepsilon^v z)) \cos \alpha + i \sin \alpha] - 1 \right) + 1.
\]

(24)

Let \( v = 0, 1, 2, \ldots, k - 1 \) in (24) respectively and summing them, we get

\[
\frac{z[f_{j,k}^{\lambda,m}(c, d; z)]'}{f_{j,k}^{\lambda,m}(c, d; z)} = \gamma \frac{k}{k} \sum_{v=0}^{k-1} \left( e^{-i\alpha} [h(\omega(\varepsilon^v z)) \cos \alpha + i \sin \alpha] - 1 \right) + 1.
\]

(25)

From the equality (25), we get

\[
\frac{[f_{j,k}^{\lambda,m}(c, d; z)]'}{f_{j,k}^{\lambda,m}(c, d; z)} - \frac{1}{z} = \frac{\gamma}{k} \sum_{v=0}^{k-1} \frac{e^{-i\alpha} [h(\omega(\varepsilon^v z)) \cos \alpha + i \sin \alpha] - 1}{z}.
\]

Integrating this equality, we get

\[
\log \left\{ \frac{f_{j,k}^{\lambda,m}(c, d; z)}{z} \right\} = \frac{k}{2} \sum_{v=0}^{k-1} \int_0^{e^v z} \frac{e^{-i\alpha} [h(\omega(\varepsilon^v \zeta)) \cos \alpha + i \sin \alpha] - 1}{\zeta} \, d\zeta,
\]

\[
= \frac{k}{2} \sum_{v=0}^{k-1} \int_0^{e^v z} \frac{e^{-i\alpha} [h(\omega(\varepsilon^v t)) \cos \alpha + i \sin \alpha] - 1}{t} \, dt,
\]

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or equivalently,

\[ f_{\lambda,m}^{j,k}(c,d;z) = z \exp \left\{ \frac{\gamma}{k} \sum_{v=0}^{k-1} \int_0^{\varepsilon v z} e^{-i\alpha} \left[ h(\omega(\varepsilon vt)) \cos \alpha + i \sin \alpha \right] dt \right\}. \]

This completes the proof of Theorem 2.

**Theorem 3.** Let \( f \in S_{j,k}^{\lambda,m} (c,d,\alpha,\beta,\gamma) \). Then we have

\[ D^m_c, d f(z) = \int_0^z \exp \left\{ \frac{\gamma}{k} \sum_{v=0}^{k-1} \int_0^{e^v \zeta} e^{-i\alpha} \left[ h(\omega(\zeta t)) \cos \alpha + i \sin \alpha \right] dt \right\} \cdot \left( \gamma (e^{-i\alpha} [h(\omega(\zeta)) \cos \alpha + i \sin \alpha] - 1 + 1) \right) d\zeta, \] (26)

where \( \omega(z) \) is analytic in \( U \) and \( \omega(0) = 0, |\omega(z)| < 1 \).

3. **Inclusion Properties of the Class \( S_{j,k}^{\lambda,m} (c,d,\alpha,\beta,\gamma) \)**

In this section, we will prove inclusion property associated with generalized Bernardi integral operator given by

\[ L_\mu[f](z) = \mu + 1 \int_0^z \frac{f(t)}{z^{\mu}} dt, \quad (f \in A, \mu > -1). \] (27)

To establish our results in this section, We need the following Lemmas.

**Lemma 4.** Let \( h \) be convex in \( U \), with \( \text{Re}[\beta h(z) + \gamma] > 0 \). If \( p(z) \) is analytic in \( U \) with \( p(0) = h(0) \), then

\[ p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} < h(z) \quad \implies \quad p(z) < h(z). \]

**Lemma 5.** Let \( h \) be convex in \( U \), with \( \text{Re}[\beta h(z) + \gamma] > 0 \). If \( f \in A \) and \( F \) is given by (27), then

\[ \frac{zf'(z)}{f(z)} < h(z) \quad \implies \quad \frac{zf'(z)}{F(z)} < h(z). \]

**Theorem 6.** Let \( 0 \leq \lambda \leq 1 \) and \( h(z) \) be convex univalent function, then

\[ S_{j,k}^{\lambda,m+1} (c,d,\alpha,\beta,\gamma) \subset S_{j,k}^{\lambda,m} (c,d,\alpha,\beta,\gamma). \]
Proof. Let \( f \in S_{j,k}^{\lambda,m}(c, d, \alpha, \beta, \gamma) \) and set

\[
l(z) = \frac{z[D_{\lambda}^m(c, d)f(z)]'}{f_{j,k}^{\lambda,m}(c, d; z)}, \quad m(z) = \frac{z[f_{j,k}^{\lambda,m}(c, d; z)]'}{f_{j,k}^{\lambda,m}(c, d; z)},
\]

(28)

we observe that \( l(z) \) and \( m(z) \) are analytic in \( \mathcal{U} \) with \( l(0) = m(0) = 1 \). Then by applying (12) in \( l(z) \), we obtain

\[
l(z)f_{j,k}^{\lambda,m}(c, d; z) = \frac{1}{\lambda}D_{\lambda}^{m+1}(c, d)f(z) - \frac{(1 - \lambda)}{\lambda}D_{\lambda}^m(c, d)f(z).
\]

(29)

Differentiating both sides of equation (29) with respect to \( z \), we get after simple computation

\[
z l'(z) + \left( \frac{1 - \lambda}{\lambda} + \frac{z[f_{j,k}^{\lambda,m}(c, d; z)]'}{f_{j,k}^{\lambda,m}(c, d; z)} \right) l(z) = \frac{1}{\lambda} \frac{z[D_{\lambda}^{m+1}(c, d)f(z)]'}{f_{j,k}^{\lambda,m}(c, d; z)}.
\]

(30)

Using the relation between (12) and (16), we can easily deduce that

\[
z[f_{j,k}^{\lambda,m}(c, d; z)]' + \frac{(1 - \lambda)}{\lambda}f_{j,k}^{\lambda,m}(c, d; z) = \frac{1}{\lambda}f_{j,k}^{\lambda,m+1}(c, d; z).
\]

(31)

Using (31) in (30), we have

\[
l(z) + z l'(z) \left( \frac{1 - \lambda}{\lambda} + \frac{z[f_{j,k}^{\lambda,m}(c, d; z)]'}{f_{j,k}^{\lambda,m}(c, d; z)} \right)^{-1} = \frac{z[D_{\lambda}^{m+1}(c, d)f(z)]'}{f_{j,k}^{\lambda,m+1}(c, d; z)}.
\]

From the definition of \( f \in S_{j,k}^{\lambda,m}(c, d, \alpha, \beta, \gamma) \), we have

\[
l(z) + \frac{z l'(z)}{(1 - \lambda) + m(z)} \prec \gamma \left( e^{-i\alpha} [h(z) \cos \alpha + i \sin \alpha] - 1 \right) + 1.
\]

(32)

In view of Lemma 5, the assertion of the Theorem would follow once we prove that \( m(z) \prec \gamma \left( e^{-i\alpha} [h(z) \cos \alpha + i \sin \alpha] - 1 \right) + 1, (z \in \mathcal{U}). \)

It follows from \( m(z) \) and (31) that

\[
\frac{(1 - \lambda)}{\lambda} + m(z) = \frac{f_{j,k}^{\lambda,m+1}(c, d; z)}{\lambda f_{j,k}^{\lambda,m}(c, d; z)}.
\]

(33)

By logarithmical differentiation of equation (33), we get

\[
m(z) + \frac{zm'(z)}{(1 - \lambda) + \lambda m(z)} = \frac{z[f_{j,k}^{\lambda,m+1}(c, d; z)]'}{f_{j,k}^{\lambda,m+1}(c, d; z)}.
\]

(34)
Using Theorem 1 in equality (34), we have
\[ m(z) + \frac{zm'(z)}{(1 - \lambda) + \lambda m(z)} \prec \gamma \left( e^{-i\alpha} [h(z) \cos \alpha + i \sin \alpha] - 1 \right) + 1, \quad (z \in U). \] (35)

In view of Lemma (4), we deduce that
\[ m(z) \prec \gamma \left( e^{-i\alpha} [h(z) \cos \alpha + i \sin \alpha] - 1 \right) + 1. \]

This implies that
\[ S_{\lambda,m}^{\lambda+1}(c, d, \alpha, \beta, \gamma) \subset S_{\lambda,m}^{\lambda}(c, d, \alpha, \beta, \gamma). \]

**Theorem 7.** Let \( f \in A \) and \( F = L_{\mu}[f] \), where \( L_{\mu}[f] \) is defined by (27). If \( f \in S_{\lambda,m}^{\lambda}(c, d, \alpha, \beta, \gamma) \) then \( F \in S_{\lambda,m}^{\lambda}(c, d, \alpha, \beta, \gamma) \).

**Proof.** From the definition of \( F \) and the linearity of the operator \( D_{\lambda}^m(c, d)f(z) \), we have
\[ z(D_{\lambda}^m(c, d)L_{\mu}[f](z))' = (\mu + 1)(D_{\lambda}^m(c, d)f(z)) - \mu(D_{\lambda}^m(c, d)L_{\mu}[f](z)). \] (36)

From (36), we have
\[ (\mu + 1)f_{\lambda,m}^{\lambda,m}(c, d; z) = \mu F_{\lambda,m}^{\lambda,m}(c, d; z) + z(F_{\lambda,m}^{\lambda,m}(c, d; z))'. \] (37)

If we let
\[ \omega(z) = \frac{z(F_{\lambda,m}^{\lambda,m}(c, d; z))'}{F_{\lambda,m}^{\lambda,m}(c, d; z)}, \]
then \( \omega \) is analytic in \( U \) and \( \omega(0) = 1 \). From (37), we observe that
\[ \mu + \omega(z) = (\mu + 1) \frac{f_{\lambda,m}^{\lambda,m}(c, d; z)}{F_{\lambda,m}^{\lambda,m}(c, d; z)}. \] (38)

Differentiating both sides of (38) with respect to \( z \), we obtain
\[ \omega(z) + \frac{z\omega'(z)}{\mu + \omega(z)} = \frac{z(f_{\lambda,m}^{\lambda,m}(c, d; z))'}{f_{\lambda,m}^{\lambda,m}(c, d; z)}. \]

By Theorem 1, we have
\[ \omega(z) + \frac{z\omega'(z)}{\mu + \omega(z)} \prec \gamma \left( e^{-i\alpha} [h(z) \cos \alpha + i \sin \alpha] - 1 \right) + 1, \]
which on using Lemma 4 implies $\omega(z) \prec \gamma \left( e^{-i\alpha} [h(z) \cos \alpha + i \sin \alpha] - 1 \right) + 1$.

Now suppose that
\[
q(z) = \frac{z(D^m_{\lambda}(c, d)')}{F_{j,k}^{\lambda,m}(c, d; z)},
\]
then $q(z)$ is analytic in $U$, with $q(0) = 1$, and it follows from (36) that
\[
F_{j,k}^{\lambda,m}(c, d; z)q(z) = (\mu + 1)(D^m_{\lambda}(c, d)f(z)) - \mu D^m_{\lambda}(c, d)F(z). \tag{39}
\]
Differentiating both sides of (39), we get
\[
zq'(z) + (\mu + \omega(z))q(z) = (\mu + 1) \frac{z(D^m_{\lambda}(c, d)f(z))'}{F_{j,k}^{\lambda,m}(c, d; z)}. \tag{40}
\]
Now from (38) and (40), we can deduce that
\[
q(z) + \frac{zq'(z)}{\mu + \omega(z)} = \frac{z(D^m_{\lambda}(c, d)f(z))'}{F_{j,k}^{\lambda,m}(c, d; z)} \prec \gamma \left( e^{-i\alpha} [h(z) \cos \alpha + i \sin \alpha] - 1 \right) + 1.
\]
Hence an application of Lemma 5 yields
\[
q(z) \prec \gamma \left( e^{-i\alpha} [h(z) \cos \alpha + i \sin \alpha] - 1 \right) + 1,
\]
which shows that $F \in S_{j,k}^{\lambda,m}(c, d, \alpha, \beta, \gamma)$.

References


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