EXACT CONTROLLABILITY FOR THE WAVE PROBLEM WITH ROBIN CONDITIONS ON AN $\varepsilon$-PERIODIC DOMAIN

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Abstract. The paper presents the study of the exact controllability on an $\varepsilon$-periodic domain lying along two directions. The exact control is applied on a part of the boundary domain, in the case of the wave problem with Robin conditions. The result is a plane wave problem, with convection term end exactly controlled by a control which represents a combination between the limit of the initial control and the convection of the limit.

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1. Introduction
The article studies in the homogenization of a wave problem with Robin condition controlled by an exact intern control exerted on a part of the border of the structure. The problem was studied on a fixed domain in [3]. The structure is three-dimensional, rectangular type, denoted by $\Omega$, with the inferior base fixed in the plane $XOY$ (or $X_1OX_2$) and it is consisted of deformable solid. In the interior of the structure we have a parallelepiped which has the same median plan with the initial parallelepiped and in which are distributed empty spheres (holes) with period $\varepsilon$, but only following the directions $OX_1$ and $OX_2$. The thickness of the initial parallelepiped is $k\varepsilon$ ($k > 0$) and the thickness of the included parallelepiped is $hk\varepsilon$ ($0 < h < 1$), we will denote by $\Gamma_\varepsilon^+$ - the upper face and $\Gamma_\varepsilon^-$ - the base – the lower face. The domain which is occupied by the material is denoted by $\Omega_\varepsilon$ and it is an $\varepsilon$ periodically perforated domain following the directions $OX_1$ and $OX_2$ only in the band size $hk\varepsilon$. The domain is similarly to the domain from [5]. On the cover $\Gamma_\varepsilon^+$ is applied a force $v_\varepsilon$ which determines oscillations in whole structure $\Omega_\varepsilon$, $v_\varepsilon$ satisfying the exact control condition for $\Omega_\varepsilon$. We made the construction of the control $v_\varepsilon$ using the HUM method introduced by Lions in [4]. For the homogenization of the wave problem we applied the dilatation method and the two-scale convergence method.
2. The Statement of Problem of the Free Waves with Robin Conditions

First, we made dilatation \( z = \frac{x_1}{k_\varepsilon} \) that transforms the partial perforated domain \( \Omega_\varepsilon \) into \( \Omega^*_\varepsilon \), the domain where the base is in the plan \( X_1OX_2 \), the superior cover \( \Gamma_\varepsilon^+ \) is transformed into \( \Gamma^+ \), the thickness of the structure is 1, and the middle parallelepiped has the thickness \( h \).

We consider the domain \( \Omega^*_\varepsilon \) covered with the grid \( \varepsilon Y^* \), where \( Y^* \) is the periodicity cell, definite by \( Y^* = Y \setminus T \), where \( Y = (0,1)^3 \) is the representative cell and \( T \) is the hole from the interior of \( Y \), transformed from the initial sphere with the dilatation \( z = \frac{x_1}{k_\varepsilon} \). We denote by \( S_{h^+\varepsilon}^+ \) the covers of \( Y^* \). Initial, the cell \( Y^* \) is distributed in the parallelepiped \( \Omega \) with the period \( \varepsilon \).

Now, we consider the wave problem on \( \Omega_\varepsilon \)

\[
\begin{cases}
    u''_\varepsilon - \Delta u_\varepsilon + qu_\varepsilon = 0_\varepsilon & \text{in } \Omega_\varepsilon \times (0,T) \\
    \frac{\partial u_\varepsilon}{\partial v} + au_\varepsilon = 0 & \text{on } \Gamma_\varepsilon^+ \times (0,T) \\
    \frac{\partial u_\varepsilon}{\partial v} = 0 & \text{on } (\partial T_\varepsilon \cup \partial \Omega_\varepsilon^\infty) \times (0,T) \\
    u_\varepsilon = 0 & \text{on } \Gamma_\varepsilon^- \\
    u_\varepsilon(0) = u^0_\varepsilon, \ u'_\varepsilon(0) = u^1_\varepsilon & \text{in } \Omega_\varepsilon
\end{cases}
\] (1)
where $T_{\varepsilon} = \varepsilon T$ such that $T_{\varepsilon} \cap \partial \Omega = \emptyset$, $\partial \Omega_{\varepsilon}^{\infty}$ is the lateral border of $\Omega_{\varepsilon}$.

We consider the following conditions satisfied:

i) $q = q \left( \frac{x_{1}}{\varepsilon}, \frac{x_{2}}{\varepsilon} \right) = q (y_{1}, y_{2})$ and $0 < m \leq q (y_{1}, y_{2}) = q (y_{0}) \leq M$ a.e. $Y^{*}$; 
$a = a \left( \frac{x_{1}}{\varepsilon}, \frac{x_{2}}{\varepsilon} \right) = a (y_{1}, y_{2}) = a (y_{0})$ with the property $0 < \alpha \leq a (y_{0}) \leq \beta$ a.e. $S_{h}^{+}$; 
$q \in L^{\infty}_{(1,2)\text{per}} (Y^{*})$, 
$a \in L^{\infty}_{(1,2)\text{per}} (S_{h}^{+})$.

ii) $(u_{0}, u_{1}) \in V_{\varepsilon} \times L^{2} (\Omega_{\varepsilon})$ where $V_{\varepsilon}$ is the Hilbert space $V_{\varepsilon} = \{ u \in H^{1} (\Omega_{\varepsilon}) : u = 0$ on $\Gamma_{\varepsilon}^{-} \}$, the norm induced by the space $H^{1} (\Omega_{\varepsilon})$, and the condition $u_{0} \in L^{2} (\Gamma_{\varepsilon}^{+})$.

After the dilatation operation, we multiply the first equation of the system (1) by $u'_{\varepsilon}$, we integrate by parts on $\Omega_{\varepsilon} \times (0, T)$ and we obtain

\[
\frac{1}{2} \int_{0}^{T} \frac{d}{dt} \int_{\Omega_{\varepsilon}} (u'_{\varepsilon})^{2} dx_{\alpha} dz dt + \\
+ \frac{1}{2} \int_{0}^{T} \frac{d}{dt} \int_{\Omega_{\varepsilon}} \left[ \frac{\partial u_{\varepsilon}}{\partial x_{\alpha}} \cdot \frac{\partial u_{\varepsilon}}{\partial x_{\alpha}} + \frac{1}{(k\varepsilon)^{2}} \left( \frac{\partial u_{\varepsilon}}{\partial z} \right)^{2} \right]^{2} dx_{\alpha} dz dt + \\
+ \frac{1}{2} \int_{0}^{T} \frac{d}{dt} \int_{\Omega_{\varepsilon}} \mu \left( \frac{x_{\alpha}}{\varepsilon} \right) (u_{\varepsilon})^{2} dx_{\alpha} dz dt + \\
+ \frac{1}{2} \int_{0}^{T} \frac{d}{dt} \int_{\Gamma_{\varepsilon}^{+}} a \left( \frac{x_{\alpha}}{\varepsilon} \right) (u_{\varepsilon})^{2} d x_{\sigma} = 0 \tag{2}
\]

and we denote the energy of the system by:

\[
E_{\alpha} (t) = \frac{1}{2} \int_{\Omega_{\varepsilon}} (u'_{\varepsilon})^{2} dx_{\alpha} dz + \frac{1}{2} \int_{\Omega_{\varepsilon}} \left[ \frac{\partial u_{\varepsilon}}{\partial x_{\alpha}} \cdot \frac{\partial u_{\varepsilon}}{\partial x_{\alpha}} + \frac{1}{(k\varepsilon)^{2}} \left( \frac{\partial u_{\varepsilon}}{\partial z} \right)^{2} \right] dx_{\alpha} dz + \\
+ \frac{1}{2} \int_{\Omega_{\varepsilon}} q \left( \frac{x_{\alpha}}{\varepsilon} \right) (u_{\varepsilon})^{2} dx_{\alpha} dz + \frac{1}{2} \int_{\Gamma_{\varepsilon}^{+}} a \left( \frac{x_{\alpha}}{\varepsilon} \right) (u_{\varepsilon})^{2} d x_{\sigma} = \frac{1}{2} \int_{\Gamma_{\varepsilon}^{+}} a \left( \frac{x_{\alpha}}{\varepsilon} \right) (u_{\varepsilon})^{2} d x_{\sigma} (x_{\alpha})
\]
so, the relation (2) implies

\[ E_u(T) = E_u(0) = \]

\[ = \frac{1}{2} \|u_1^\varepsilon\|_{L^2(\Omega_\varepsilon^1)}^2 + \frac{1}{2} \|u_0^\varepsilon\|_{V_\varepsilon}^2 + \frac{1}{2} \int_{\Omega_\varepsilon^1} q \left( \frac{x_\alpha}{\varepsilon} \right) (u_0^\varepsilon)^2 \, dx_\alpha dz + \]

\[ + \frac{1}{2} \int_{\Gamma^+} a \left( \frac{x_\alpha}{\varepsilon} \right) (u_0^\varepsilon)^2 \, d\sigma^\varepsilon (x_\alpha) \leq C \|u_0^\varepsilon\|_{V_\varepsilon}^2 + \frac{1}{2} \|u_1^\varepsilon\|_{L^2(\Omega_\varepsilon^1)}^2. \]

We use the conservation of the energy and the conditions i), ii), and we obtain

\[ E_u(t) \leq C \]

so, we get

\[ \|u_1^\varepsilon\|_{L^2(\Omega_\varepsilon^1)} \leq C, \quad \|u_\varepsilon\|_{V_\varepsilon} \leq C, \quad \|u_\varepsilon\|_{L^2(\Gamma^+)} \leq C \]

\[ \left( \|u_\varepsilon\|_{V_\varepsilon} = \int_{\Omega_\varepsilon^1} \left[ \frac{\partial u_\varepsilon}{\partial x_\alpha} \frac{\partial u_\varepsilon}{\partial x_\alpha} + \frac{1}{(k\varepsilon)^2} \left( \frac{\partial u_\varepsilon}{\partial z} \right)^2 \right] \, dx_\alpha dz \right) \]

which implies the next two-scale convergences:

\[ \begin{cases} 
  u_\varepsilon \xrightarrow{2s} u (x_\alpha) \in H^1 (\Gamma_h^+) , & u'_\varepsilon \xrightarrow{2s} u' (x_\alpha) \in H^{-1} (\Gamma_h^+) \\
  \nabla u_\varepsilon \xrightarrow{2s} \nabla_{x_\alpha} u (x_\alpha) + \nabla_{y_\alpha} U (y_\alpha, z) + k^{-1} \nabla_{z} U (y_\alpha, z) \end{cases} \]

where

\[ U \in L^2 \left( 0, T; H^1(1,2)_{per} (Y) / \mathbb{R} \right) \]

and from ii) we obtain

\[ u_0^\varepsilon \xrightarrow{2s} \frac{u_0(x_\varepsilon)}{\text{meas} Y^\varepsilon} , \quad u_1^\varepsilon \xrightarrow{2s} \frac{u_1(x_\varepsilon)}{\text{meas} Y^\varepsilon}. \]

3. The Homogenization of the Free Waves Problem with Robin Conditions

For problem (1) we apply the two-scale convergence method [1] and we find the plan hyperbolic limit problem:
\[(measY^*) u'' (x_\alpha) -\]
\[- \partial_{xx_\alpha} \left( A_{\alpha, \beta} \frac{\partial u}{\partial x_\beta} + b_{\alpha} \frac{\partial u}{\partial x_\alpha} + \lambda u \right) + c_{\alpha} (x_\alpha) = 0 \text{ in } \Gamma^+_h \times (0, T), \]
\[u (x_\alpha) = 0 \text{ on } \partial \Gamma^+_h \times (0, T), \]
\[u (0) = \frac{u^0}{measY^*}, \quad u' (0) = \frac{u^1}{measY^*} \text{ in } \Gamma^+, \]
where
\[y = (y_\alpha, z)\]

\[A_{\alpha, \beta} = \int_{Y^*} \frac{\partial (y_\alpha + \chi_\alpha (y))}{\partial \gamma} \cdot \frac{\partial (y_\beta + \chi_\beta (y))}{\partial \gamma} \, dy + \frac{1}{k^2} \int_{Y^*} \frac{\partial \chi_\alpha}{\partial z} (y) \cdot \frac{\partial \chi_\beta}{\partial z} (y) \, dy\]

\[b_{\alpha} = - \int_{Y^*} \left[ \frac{\partial \gamma}{\partial y_\alpha} (y) + \frac{1}{k} \frac{\partial \gamma}{\partial z} (y) \right] \, dy + \int_{S^+_h} a (y_\alpha) \cdot \chi_\alpha (y_\alpha, 1) \, d\sigma (y_\alpha),\]

\[\lambda = \int_{Y^*} q (y_\alpha) \, dy + \int_{S^+_h} a (y_\alpha) \cdot \gamma (y_\alpha, 1) \, d\sigma (y_\alpha),\]

where the correctors \(\chi_\beta (y), \, \gamma (y) \in H^{1, 2}_{(1, 2)_{\text{per}}} (Y), \, (\beta = 1, 2)\) verifies the weak microscopic problems

\[\int_{Y^*} \frac{\partial (y_\beta + \chi_\beta (y))}{\partial y_\alpha} \cdot \frac{\partial q}{\partial y_\alpha} \, dy + \frac{1}{k^2} \int_{Y^*} \frac{\partial \chi_\beta}{\partial z} (y) \cdot \frac{\partial q}{\partial z} (y) \, dy = 0,\]

\[\int_{Y^*} \left[ \frac{\partial \gamma}{\partial y_\alpha} (y) \cdot \frac{\partial q}{\partial y_\alpha} (y) + \frac{1}{k^2} \frac{\partial \gamma}{\partial z} (y) \cdot \frac{\partial q}{\partial z} (y) \right] \, dy + \int_{S^+_h} a (y_\alpha) \cdot q (y_\alpha, 1) \, d\sigma (y_\alpha) = 0,\]

\[\forall q \in H^{1, 2}_{(1, 2)_{\text{per}}} (Y^*/\mathbb{R}).\]
4. The HUM Method for the Construction of the Exact Control of the Problem (1)

We consider the system

\[
\begin{align*}
\phi'' - \Delta \phi + q \phi &= 0 \text{ in } \Omega \times (0, T), \\
\frac{\partial \phi}{\partial t} + a \phi &= 0 \text{ on } \Gamma^+ \times (0, T), \\
\phi(0) &= \phi^0, \phi'(0) = \phi^1,
\end{align*}
\]

where \((\phi^0, \phi^1) \in L^2(\Omega) \times V_\varepsilon^\prime\), \(\|\phi^0\|_{L^2(\Omega)} \leq C\), \(\|\phi^1\|_{V_\varepsilon^\prime} \leq C\), and the retrograde system:

\[
\begin{align*}
y'' - \Delta y + qu &= 0 \text{ in } \Omega \times (0, T), \\
\frac{\partial y}{\partial t} + ay &= -\phi \text{ on } \Gamma^+ \times (0, T), \\
y(0) &= \phi^0, y'(0) = \phi^1,
\end{align*}
\]

and we consider the application

\[
\Lambda_\varepsilon : F_\varepsilon \to F_\varepsilon^\prime \quad \Lambda_\varepsilon (\phi^0, \phi^1) = (y^0, -y^1) \Rightarrow
\]

\[
\langle \Lambda_\varepsilon (\phi^0, \phi^1), (\phi^0, \phi^1) \rangle_{F_\varepsilon^\prime, F_\varepsilon} = \int_{\Omega} [y^0(\phi^0 - y^1)] d\Omega,
\]

where \(F_\varepsilon = L^2(\Omega) \times V_\varepsilon^\prime\) and \(F_\varepsilon^\prime = L^2(\Omega) \times V_\varepsilon\).

We multiply the first equation from the system (4) by \(\phi\), we integrate by parts two times on \(\Omega \times (0, T)\) and we obtain

\[
0 = \int_0^T \int_{\Omega} (y'' - \Delta y + qu) \phi d\Omega dt =
\]

\[
= \left[ \int_{\Omega} (y' \phi - y \phi') dx \right]_0^T - \int_0^T \int_{\Gamma^+} (\frac{\partial y}{\partial t} \phi - y \frac{\partial \phi}{\partial t}) d\sigma (x) dt +
\]

\[
+ \int_0^T \int_{\Omega} y'' (\phi - \Delta \phi + q \phi) dx dt =
\]
Using the relation (5) we have:

$$
\langle \Lambda_\varepsilon (\phi_\varepsilon^0, \phi_\varepsilon^1), (\phi_\varepsilon^0, \phi_\varepsilon^1) \rangle_{F^\varepsilon, F^\varepsilon} = \int_0^T \int_{\Gamma^+_\varepsilon} \phi_\varepsilon^2 d\sigma^\varepsilon (x) dt =

= \| \phi_\varepsilon \|^2_{L^2(0,T;L^2(\Gamma^+_\varepsilon))} = \| (\phi_\varepsilon^0, \phi_\varepsilon^1) \|^2_{F^\varepsilon}
$$

we deduce that

$$
\| \Lambda_\varepsilon (\phi_\varepsilon^0, \phi_\varepsilon^1) \|_{F^\varepsilon} = \| (\phi_\varepsilon^0, \phi_\varepsilon^1) \|_{F^\varepsilon} = \left( \| \phi_\varepsilon^0 \|^2_{L^2(\Omega_\varepsilon)} + \| \phi_\varepsilon^1 \|^2_{V^\prime_\varepsilon} \right)^{1/2} \leq C
$$

(7)

it means that $\Lambda_\varepsilon$ is bounded and from relation (6) results that we can apply Lax-Milgram, so that $\Lambda_\varepsilon$ is an isomorphism from $F^\varepsilon$ to $F^\varepsilon$.

Now, we consider the system (1) to which we attach an application $v_\varepsilon = -\phi_\varepsilon$ on $\Gamma^+_\varepsilon$ and we have

$$
\begin{cases}
    u_\varepsilon'' - \Delta u_\varepsilon + qu_\varepsilon = 0 \text{ in } \Omega_\varepsilon \times (0,T), \\
    \frac{\partial u_\varepsilon}{\partial \nu} + au_\varepsilon = v_\varepsilon \text{ on } \Gamma^+_\varepsilon \times (0,T), \\
    u_\varepsilon = 0 \text{ on } \Gamma^-_\varepsilon \times (0,T), \\
    \frac{\partial u_\varepsilon}{\partial \nu} = 0 \text{ on } (\partial T_\varepsilon \cup \partial \Omega_\varepsilon^\infty) \times (0,T), \\
    u_\varepsilon (0) = u_\varepsilon^0, u_\varepsilon^\prime (0) = u_\varepsilon^1 \text{ in } \Omega_\varepsilon.
\end{cases}
$$

(8)

But $\Lambda_\varepsilon$ is an isomorphism, so result

$$
y'_\varepsilon (0) = u_\varepsilon^1, \quad y_\varepsilon (0) = u_\varepsilon^0
$$

and because $v_\varepsilon = -\phi_\varepsilon$ we observe that $y_\varepsilon$ is the solution of the problem (8) which has unique solution, so

$$
y_\varepsilon = u_\varepsilon \Rightarrow u_\varepsilon (T) = u_\varepsilon^\prime (T) = 0
$$

and the system (1) accepts an exact control $\varepsilon \in L^2 (0,T;L^2 (\Gamma^+_\varepsilon))$. 

5. The Limit of the Exact Control

Because $v_\varepsilon = -\phi_\varepsilon$, it is enough to study the convergence of $\phi_\varepsilon$. The first equation of the system (3) is multiplied with $\phi_\varepsilon$, then we integrate it by parts on $\Omega \times (0, T)$, we take into account the conditions satisfied by $\phi_\varepsilon^0$ and $\phi_\varepsilon^1$, we find like in section 1:

$$\|\phi_\varepsilon\|_{L^2(0,T;L^2(\Omega_\varepsilon))} \leq C$$

and from equations (6), (7), we find

$$\|\phi_\varepsilon\|_{L^2(0,T;L^2(\Gamma^+))} \leq C$$

and from relation (3) we get the estimation

$$\|\phi_\varepsilon^0\|_{L^2(\Omega_\varepsilon)} \leq C.$$  

From all these relations we obtain the two-scale convergences

$$\phi_\varepsilon \xrightarrow{2s} \phi(x_\alpha), \quad \phi_\varepsilon^0 \xrightarrow{2s} \phi^0(x_\alpha)(\text{meas}Y^*).$$

Because $\phi_\varepsilon^1$ isn’t a regular function, we apply the regularization method of a problem (3), resulting to problem with regular conditions, we compute the limit for it (like for problem (1)) and finally we obtain the next limit problem for the control limit:

$$\begin{cases}
\left(\text{meas}Y^*\right) \phi''(x_\alpha) - \frac{\partial}{\partial x_\alpha} \left( A_{\alpha\beta} \frac{\partial \phi}{\partial x_\beta}(x_\alpha) \right) + \\
= b_{\alpha} \frac{\partial \phi}{\partial x_\alpha}(x_\alpha) + \lambda \phi(x_\alpha) = 0 \quad \text{in } \Gamma^+ \times (0, T), \\
\phi = 0 \quad \text{on } \Gamma^+ \times (0, T), \\
\phi(0) = \frac{\phi^0}{\text{meas}Y^*}, \phi'(0) = \frac{\phi^{1,*}}{\text{meas}Y^*} \quad \text{in } \Gamma^+
\end{cases}$$

where $\phi^{1,*}(x_\alpha)$ is equal with

$$\phi^{1,*}(x_\alpha) = \frac{\partial g^*}{\partial x_1}(x_\alpha) + \frac{\partial g^*}{\partial x_2}(x_\alpha)$$

where we have the convergence

$$g^\varepsilon \xrightarrow{2s} g^*(x_\alpha), \quad \beta = 1, 2$$

and

$$g^\varepsilon_{\beta}(x_\alpha, z) = \frac{\partial \rho_\varepsilon}{\partial x_\alpha} \cdot \frac{\partial X_\varepsilon^{\beta}}{\partial x_\alpha}(x_\alpha) + \frac{1}{k^2} \frac{\partial \rho_\varepsilon}{\partial z}(x_\alpha, z) \cdot \frac{\partial X_\varepsilon^{\beta}}{\partial z}(x_\alpha, z), \quad \beta = 1, 2.$$
with \( \rho_\varepsilon \) is the solution of the elliptical problem a little regular:

\[
\begin{aligned}
&\left\{
- \left[ \frac{\partial^2 \rho_\varepsilon}{\partial x_1^2} + \frac{\partial^2 \rho_\varepsilon}{\partial x_2^2} + \frac{1}{k^2} \frac{\partial^2 \rho_\varepsilon}{\partial z^2} \right] + \\
&+ q \left( \frac{x_\alpha}{\varepsilon} \right) \rho_\varepsilon \left( x_\alpha, z \right) = -\phi_\varepsilon^1 \text{ in } \Omega^*_\varepsilon, \\
&\frac{\partial \rho_\varepsilon}{\partial n} = 0 \text{ on } (\partial T_\varepsilon \cup \partial \Omega^\infty_\varepsilon), \\
&\rho_\varepsilon = 0 \text{ on } \Gamma^-, \\
&\frac{\partial \rho_\varepsilon}{\partial n} + a \left( \frac{x_\alpha}{\varepsilon} \right) \rho_\varepsilon \left( x_\alpha, z \right) = 0 \text{ on } \Gamma^+.
\end{aligned}
\]

This regularization method is in [2], and \( \phi^{1,*} \) is obtained in [6].

Finally, we obtain a macroscopic problem of controlled waves:

\[
\begin{aligned}
&\left\{
\left( measY^* \right) u'' - \frac{\partial}{\partial x_\alpha} \left( A_{\alpha \beta} \frac{\partial u}{\partial x_\beta} \right) + \\
&+ b_\alpha \frac{\partial u}{\partial x_\alpha} + \lambda u = F(x_\alpha) \text{ in } \Gamma^+ \times (0, T), \\
&u = 0 \text{ on } \partial \Gamma^+ \times (0, T), \\
&u(0) = \frac{u^0}{measY^*}, u'(0) = \frac{u^1}{measY^*} \text{ in } \Gamma^+.
\end{aligned}
\]

where the control of the limit problem is

\[
F(x_\alpha) = \left[ \int_{S_h^+} \chi^\beta (y_\alpha, 1) \, d\sigma(y_\alpha) \right] \cdot \frac{\partial v}{\partial x_\beta}(x_\alpha) + \left[ \int_{S_h^+} \gamma (y_\alpha, 1) \, d\sigma(y_\alpha) \right] \cdot v(x_\alpha)
\]

where \( v = -\phi \in L^2 \left( 0, T; L^2 \left( \Gamma^+ \right) \right) \).

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