SOME SANDWICH-TYPE RESULTS FOR $\phi$-LIKE FUNCTIONS

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Abstract. Using the technique of differential subordination, we here obtain certain results for $\phi$-like, starlike and close-to-convex functions.

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1. Introduction

Let $\mathcal{H}$ be the class of functions analytic in $E = \{z : |z| < 1\}$ and $\mathcal{H}[a,n]$ be the subclass of $\mathcal{H}$ consisting functions of the form

$$f(z) = a + a_nz^n + a_{n+1}z^{n+1} + \cdots.$$ 

Let $\mathcal{A}$ be the subclass of $\mathcal{H}$ consisting of functions $f$, analytic in the open unit disk $E = \{z : |z| < 1\}$ and normalized by the conditions $f(0) = f'(0) - 1 = 0$. A function $f \in \mathcal{A}$ is said to be starlike of order $\beta$, $0 \leq \beta < 1$, if and only if

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > \beta, \quad z \in E.$$ 

The class of such functions is denoted by $\mathcal{S}^*(\beta)$. Note that $\mathcal{S}^*(0) = \mathcal{S}^*$, the class of univalent starlike functions.

A function $f \in \mathcal{A}$ is said to be close-to-convex in $E$ if it satisfies the condition

$$\Re\left(\frac{zf'(z)}{g(z)}\right) > 0, \quad z \in E, \quad \text{for } g \in \mathcal{S}^*.$$ 

The class of close-to-convex functions is denoted by $\mathcal{C}$. Noshiro [2] and Warchawski [6] independently proved in 1934-35 that $f$ is close-to-convex if

$$\Re(f'(z)) > 0.$$
Let $\Phi : \mathbb{C}^2 \times \mathbb{E} \to \mathbb{C}$ be an analytic function, $p$ be an analytic function in $\mathbb{E}$ with $(p(z), zp'(z); z) \in \mathbb{C}^2 \times \mathbb{E}$ for all $z \in \mathbb{E}$ and $h$ be univalent in $\mathbb{E}$. Then the function $p$ is said to satisfy first order differential subordination if

$$\Phi(p(z), zp'(z); z) \prec h(z), \quad \Phi(p(0), 0; 0) = h(0). \quad (1)$$

A univalent function $q$ is called a dominant of the differential subordination $(1)$ if $p(0) = q(0)$ and $p \prec q$ for all $p$ satisfying $(1)$. A dominant $\tilde{q}$ that satisfies $\tilde{q} \prec q$ for all dominants $q$ of $(1)$ is said to be the best dominant of $(1)$.

Let $\Psi : \mathbb{C}^2 \times \mathbb{E} \to \mathbb{C}$ be analytic and univalent in domain $\mathbb{C}^2 \times \mathbb{E}$, $h$ be analytic in $\mathbb{E}$, $p$ be analytic and univalent in $\mathbb{E}$, with $(p(z), zp'(z); z) \in \mathbb{C}^2 \times \mathbb{E}$ for all $z \in \mathbb{E}$. Then $p$ is called a solution of the first order differential superordination if

$$h(z) \prec \Psi(p(z), zp'(z); z), \quad h(0) = \Psi(p(0), 0; 0). \quad (2)$$

An analytic function $q$ is called a subordinant of the differential superordination $(2)$, if $q \prec p$ for all $p$ satisfying $(2)$. A univalent subordinant $\tilde{q}$ that satisfies $q \prec \tilde{q}$ for all subordinates $q$ of $(2)$ is said to be the best subordinant of $(2)$.

The function $f \in \mathcal{A}$ is called $\phi$—like in the open unit disk $\mathbb{E}$, if

$$\Re \left( \frac{zf'(z)}{\phi(f(z))} \right) > 0, \quad z \in \mathbb{E},$$

where $\phi$ is analytic in a domain containing $f(\mathbb{E})$, $\phi(0) = 0 = \phi'(0) - 1$ and $\phi(w) \neq 0$ for $w \in f(\mathbb{E}) \setminus \{0\}$. This concept was first introduced by Brickman [1] and he established that a function $f \in \mathcal{A}$ is univalent if and only if $f$ is $\phi$—like for some $\phi$.

Using the concept of differential subordination Ruscheweyh [9] introduced and studied the following more general class of $\phi$—like functions:

Let $\phi$ be analytic function in the domain containing $f(\mathbb{E})$, $\phi(0) = 0 = \phi'(0) - 1$ and $\phi(w) \neq 0$ for $w \in f(\mathbb{E}) \setminus \{0\}$. Then $f \in \mathcal{A}$ is called $\phi$—like w.r.t. a univalent function $q(z)$ if

$$\frac{zf'(z)}{\phi(f(z))} \prec q(z), \quad z \in \mathbb{E}.$$

In 2005, Ravichandran et al.[10] proved the following result for $\phi$—like functions:

Let $\alpha \neq 0$ be a complex number and $q(z)$ be a convex univalent function in $\mathbb{E}$. Suppose $h(z) = \alpha q^2(z) + (1 - \alpha)q(z) + \alpha zq'(z)$ and

$$\Re \left\{ \frac{1 - \alpha}{\alpha} + 2q(z) + \left( 1 + \frac{zq''(z)}{q'(z)} \right) \right\} > 0, \quad z \in \mathbb{E}.$$

If $f \in \mathcal{A}$ satisfies

$$\frac{zf'(z)}{\phi(f(z))} \left( 1 + \frac{\alpha zf''(z)}{f'(z)} + \frac{\alpha(f'(z) - (\phi(f(z)))'}{\phi(f(z))} \right) < h(z)$$

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then
\[
\frac{zf'(z)}{\phi(f(z))} \prec q(z), \quad z \in \mathbb{E}
\]
and \(q(z)\) is best dominant.

Recently, Shanmugam et al. [5] and Ibrahim [3] also obtained the results for \(\phi\)-like functions parallel to the results of Ravichandran [10] stated above.

In the present paper, we investigate the differential operator
\[
a \frac{zf'(z)}{\phi(g(z))} + b \left( 1 + \frac{zf''(z)}{f'(z)} - \frac{z(\phi(g(z))')}{\phi(g(z))} \right),
\]
where \(f, g \in A\) and \(\phi\) is an analytic function in a domain containing \(g(\mathbb{E})\) such that \(\phi(0) = 0 = \phi'(0) - 1\) and \(\phi(w) \neq 0\) for \(w \in g(\mathbb{E}) \setminus \{0\}\), for real numbers \(a\) and \(b \neq 0\).

We, here, obtain some sufficient conditions for \(\phi\)-like, starlike and close-to-convex functions.

2. Preliminaries

We shall need the following definition and Lemmas to prove our main results.

**Definition 1.** [7, Def. 2.2h, p.21]. We denote by \(Q\) the set of functions \(p\) that are analytic and injective in \(E \setminus B(p)\), where
\[
B(p) = \left\{ \zeta \in \partial \mathbb{E} : \lim_{z \to \zeta} p(z) = \infty \right\},
\]
are such that \(p'(\zeta) \neq 0\) for \(\zeta \in \partial \mathbb{E} \setminus B(p)\).

**Lemma 1.** [7, Theorem 3.4h, p.132]. Let \(q\) be univalent in \(E\) and let \(\theta\) and \(\varphi\) be analytic in a domain \(D\) containing \(q(\mathbb{E})\), with \(\varphi(w) \neq 0\), when \(w \in q(\mathbb{E})\). Set \(Q_1(z) = zq'(z)\varphi(q(z)), h(z) = \theta[q(z)] + Q_1(z)\) and suppose that either
(i) \(h\) is convex, or
(ii) \(Q_1\) is starlike.
In addition, assume that
(iii) \(\Re \left( \frac{zh'(z)}{Q_1(z)} \right) > 0\).
If \(p\) is analytic in \(E\), with \(p(0) = q(0), p(\mathbb{E}) \subset D\) and
\[
\theta[p(z)] + zp'(z)\varphi[p(z)] \prec \theta[q(z)] + zq'(z)\varphi[q(z)],
\]
then \(p(z) \prec q(z)\) and \(q(z)\) is the best dominant.
Lemma 2. [4]. Let \( q \) be univalent in \( \mathbb{E} \) and let \( \theta \) and \( \varphi \) be analytic in a domain \( \mathbb{D} \) containing \( q(\mathbb{E}) \). Set \( Q_1(z) = zq'(z)\varphi[q(z)] \), \( h(z) = \theta[q(z)] + Q_1(z) \) and suppose that 
(i) \( Q_1 \) is starlike in \( \mathbb{E} \) and 
(ii) \( \Re \left[ \frac{\theta'(q(z))}{\varphi(q(z))} \right] > 0, \ z \in \mathbb{E}. \)
If \( p \in H[q(0), 1] \cap Q \), with \( p(\mathbb{E}) \subset \mathbb{D} \) and \( \theta[p(z)] + zp'(z)\varphi[p(z)] \) is univalent in \( \mathbb{E} \) and 
\[
\theta[q(z)] + zq'(z)\varphi[q(z)] \prec \theta[p(z)] + zp'(z)\varphi[p(z)], \ z \in \mathbb{E},
\]
then \( q(z) \prec p(z) \) and \( q(z) \) is the best subordinant.

3. Main results

Theorem 3. Let \( q, q(z) \neq 0 \) be a univalent function in \( \mathbb{E} \) and satisfies the condition 
\[
\Re \left( 1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)} \right) > \max \left\{ 0, \ -\frac{a}{b} \Re(q(z)) \right\}, \tag{3}
\]
where \( a \) and \( b(\neq 0) \) are real numbers. Let \( \phi \) be analytic function in a domain containing \( g(\mathbb{E}), \phi(0) = 0 = \phi'(0) - 1 \) and \( \phi(w) \neq 0 \) for \( w \in g(\mathbb{E}) \setminus \{0\} \). If \( f, g \in \mathcal{A}, \ \frac{zf'(z)}{\phi(g(z))} \neq 0, \ z \in \mathbb{E} \), satisfy the differential subordination 
\[
\frac{zf'(z)}{\phi(g(z))} + b \left( 1 + \frac{zf''(z)}{f'(z)} - \frac{z(\phi(g(z))')}{\phi(g(z))} \right) \prec a q(z) + b \frac{zq'(z)}{q(z)}, \tag{4}
\]
then 
\[
\frac{zf'(z)}{\phi(g(z))} \prec q(z), \ z \in \mathbb{E},
\]
and \( q(z) \) is the best dominant.

Proof. Define the function \( p(z) \) by
\[
p(z) = \frac{zf'(z)}{\phi(g(z))}.
\]
Therefore 
\[
\frac{zp'(z)}{p(z)} = 1 + \frac{zf''(z)}{f'(z)} - \frac{z(\phi(g(z))')}{\phi(g(z))}
\]
and (4) reduces to 
\[
ap(z) + b \frac{zp'(z)}{p(z)} \prec a q(z) + b \frac{zq'(z)}{q(z)}.
\]
Define $\theta$ and $\varphi$ as $\theta(w) = aw$ & $\varphi(w) = \frac{b}{w}$. Both $\theta$ and $\varphi$ are analytic in $\mathbb{C} \setminus \{0\}$ and $\varphi(w) \neq 0$, $w \in \mathbb{C} \setminus \{0\}$. Therefore $Q_1(z) = zq'(z)\varphi(q(z)) = b\frac{z q'(z)}{q(z)}$ and

$$h(z) = \theta(q(z)) + Q_1(z) = aq(z) + b\frac{z q'(z)}{q(z)}.$$

A little calculation yields

$$\frac{zQ_1(z)}{Q_1(z)} = 1 + \frac{z q''(z)}{q'(z)} - \frac{z q'(z)}{q(z)}$$

and

$$\frac{zh'(z)}{Q_1(z)} = \frac{a}{b}q(z) + 1 + \frac{z q''(z)}{q'(z)} - \frac{z q'(z)}{q(z)}.$$

In view of Condition 3, we have $Q_1(z)$ is starlike in $E$ and $\Re\left(\frac{zh'(z)}{Q_1(z)}\right) > 0$.

The proof, now, follows from the Lemma 1.

On taking $\phi(z) = z$ in Theorem 3, we have the following result:

**Theorem 4.** Let $q, q(z) \neq 0$, be a univalent function in $E$, satisfying the Condition 3 of Theorem 3 for real numbers $a, b(\neq 0)$. If $f, g \in \mathcal{A}, \frac{zf'(z)}{g(z)} \neq 0, z \in E$, satisfy the differential subordination

$$a \frac{zf'(z)}{g(z)} + b \left(1 + \frac{zf''(z)}{f'(z)} - \frac{z q'(z)}{g(z)}\right) < aq(z) + b\frac{z q'(z)}{q(z)},$$

then

$$\frac{zf'(z)}{g(z)} < q(z), \ z \in E,$$

and $q(z)$ is the best dominant.

On taking $\phi(z) = z$ and $g(z) = f(z)$ in Theorem 3, we have the following result:

**Theorem 5.** Let $q, q(z) \neq 0$ be a univalent function in $E$ and satisfies the Condition 3 of Theorem 3 for real numbers $a$ and $b(\neq 0)$. If $f \in \mathcal{A}, \frac{zf'(z)}{f(z)} \neq 0, z \in E$, satisfies

$$(a - b) \frac{zf'(z)}{f(z)} + b \left(1 + \frac{zf''(z)}{f'(z)}\right) < aq(z) + b\frac{z q'(z)}{q(z)},$$

then

$$\frac{zf'(z)}{f(z)} < q(z), \ z \in E,$$

and $q(z)$ is the best dominant.
On selecting \( a = 1 \) and \( b = \alpha \) in the Theorem 5, we get the following result for the class of \( \alpha \)-convex functions.

**Theorem 6.** Let \( \alpha \) be a non zero real number and let \( q, q(z) \neq 0 \) be a univalent function in \( E \) satisfying the Condition 3 of Theorem 3. If \( f \in A, z \in E \), satisfies

\[
(1 - \alpha) \frac{zf'(z)}{f(z)} + \alpha \left(1 + \frac{zf''(z)}{f'(z)}\right) < q(z) + \alpha \frac{zq'(z)}{q(z)},
\]

then

\[
\frac{zf'(z)}{f(z)} < q(z), \quad z \in E,
\]

and \( q(z) \) is the best dominant.

By defining \( \phi(z) = g(z) = z \) in Theorem 3, we obtain the following result:

**Theorem 7.** Let \( q, q(z) \neq 0 \) be a univalent function in \( E \) and satisfying the Condition 3 of Theorem 3 for real numbers \( a, b \neq 0 \). If \( f \in A, f'(z) \neq 0, z \in E \), satisfies the differential subordination

\[
a f'(z) + b z f''(z) \frac{1}{f'(z)} < a q(z) + b zq'(z) \frac{1}{q(z)},
\]

then

\[
f'(z) < q(z), \quad z \in E,
\]

and \( q(z) \) is the best dominant.

**Remark 1.** It is easy to verify that dominant \( q(z) = \left(\frac{1+z}{1-z}\right)^\delta, \quad 0 < \delta \leq 1 \), satisfies the Condition 3 of Theorem 3, for real numbers \( a \) and \( b(\neq 0) \). Consequently, we get:

**Theorem 8.** Let \( \phi \) be analytic function in the domain containing \( g(E) \) such that \( \phi(0) = 0 = \phi'(0) - 1 \) and \( \phi(w) \neq 0 \) for \( w \in g(E) \setminus \{0\} \). If \( f, g \in A, \frac{zf'(z)}{\phi(g(z))} \neq 0, z \in E \), and for real numbers \( a \) and \( b(\neq 0) \), satisfy

\[
a z f'(z) \frac{1}{\phi(g(z))} + b \left(1 + z f''(z) \frac{1}{f'(z)} - z \phi(g(z))' \frac{1}{\phi(g(z))}\right) < a \left(1 + \frac{z}{1-z}\right)^\delta + \frac{2b \delta z}{1-z^2},
\]

then

\[
\frac{zf'(z)}{\phi(g(z))} < \left(\frac{1+z}{1-z}\right)^\delta, \quad z \in E, \quad 0 < \delta \leq 1.
\]

On taking \( \phi(z) = z \) in above theorem, we obtain:
Corollary 9. Let $a$ and $b(\neq 0)$ are real numbers and $0 < \delta \leq 1$. If $f, g \in \mathcal{A}$, $\frac{zf'(z)}{g(z)} \neq 0$, $z \in \mathbb{E}$, satisfy

$$a \frac{zf'(z)}{g(z)} + b \left(1 + \frac{zf''(z)}{f'(z)} - \frac{zg'(z)}{g(z)}\right) < a \left(\frac{1 + z}{1 - z}\right)^\delta + \frac{2b\delta z}{1 - z^2},$$

then

$$\frac{zf'(z)}{g(z)} < \left(\frac{1 + z}{1 - z}\right)^\delta, \ z \in \mathbb{E}.$$

For $\phi(z) = z$ and $g(z) = f(z)$ in Theorem 8, we obtain the following result:

Corollary 10. Let $a$ and $b(\neq 0)$ are real numbers and $0 < \delta \leq 1$. If $f \in \mathcal{A}$, $\frac{zf'(z)}{f(z)} \neq 0$, $z \in \mathbb{E}$, satisfies the differential subordination

$$(a - b) \frac{zf'(z)}{f(z)} + b \left(1 + \frac{zf''(z)}{f'(z)}\right) < a \left(\frac{1 + z}{1 - z}\right)^\delta + \frac{2b\delta z}{1 - z^2},$$

then

$$\frac{zf'(z)}{f(z)} < \left(\frac{1 + z}{1 - z}\right)^\delta, \ z \in \mathbb{E},$$

and hence $f(z)$ is starlike.

Selecting $a = 1$ and $b = \alpha$ in above corollary, we get the following result for the class of $\alpha$-convex functions:

Corollary 11. Let $\alpha$ be a non-zero real number. If $f \in \mathcal{A}$, $\frac{zf'(z)}{f(z)} \neq 0$, $z \in \mathbb{E}$, satisfies

$$(1 - \alpha) \frac{zf'(z)}{f(z)} + \alpha \left(1 + \frac{zf''(z)}{f'(z)}\right) < \left(\frac{1 + z}{1 - z}\right)^\delta + \frac{2b\delta z}{1 - z^2},$$

then

$$\frac{zf'(z)}{f(z)} < \left(\frac{1 + z}{1 - z}\right)^\delta, \ z \in \mathbb{E}, \ 0 < \delta \leq 1.$$

Hence $f(z)$ is strongly starlike.

On taking $\phi(z) = g(z) = z$ in Theorem 8, we have:
Corollary 12. Let $a$ and $b (\neq 0)$ be real numbers. If $f \in \mathcal{A}, \ f'(z) \neq 0, \ z \in \mathbb{E}$, satisfies

$$af'(z) + bzf''(z) f'(z) \prec a \left( \frac{1 + z}{1 - z} \right)^\delta + \frac{2b \delta z}{1 - z^2},$$

then

$$f'(z) \prec \left( \frac{1 + z}{1 - z} \right)^\delta, \ z \in \mathbb{E}, \ 0 < \delta \leq 1,$$

and hence $f(z)$ is close-to-convex.

Remark 2. When we select the dominant $q(z) = e^z$, then this dominant satisfies the Condition 3 of Theorem 3 for non-zero real numbers $a$ and $b$ such that $\Re(e^z) > -\frac{b}{a}$.

Consequently, we obtain the following result:

Theorem 13. Let $a$ and $b$ be non-zero real numbers such that $\Re(e^z) > -\frac{b}{a}$ and let $\phi$ be analytic function in a domain containing $g(\mathbb{E}), \ \phi(0) = 0 = \phi'(0) - 1$ and $\phi(w) \neq 0$ for $w \in g(\mathbb{E}) \setminus \{0\}$. If $f, g \in \mathcal{A}, \ \frac{zf'(z)}{\phi(g(z))} \neq 0, \ z \in \mathbb{E}$, satisfy

$$a \frac{zf'(z)}{\phi(g(z))} + b \left( 1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{\phi(g(z))} \right) \prec ae^z + bz,$$

then

$$\frac{zf'(z)}{\phi(g(z))} \prec e^z, \ z \in \mathbb{E}.$$

On choosing $\phi(z) = z$ in above theorem, we obtain:

Corollary 14. Let $a$ and $b$ non-zero real numbers such that $\Re(e^z) > -\frac{b}{a}$. If $f, g \in \mathcal{A}, \ \frac{zf'(z)}{g(z)} \neq 0, \ z \in \mathbb{E}$, satisfy the differential subordination

$$a \frac{zf'(z)}{g(z)} + b \left( 1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{g(z)} \right) \prec ae^z + bz,$$

then

$$\frac{zf'(z)}{g(z)} \prec e^z, \ z \in \mathbb{E}.$$

On selecting $\phi(z) = z$ and $g(z) = f(z)$ in Theorem 13, we get:
Corollary 15. Let $a$ and $b$ be non-zero real numbers such that $\Re(e^z) > -\frac{b}{a}$. If $f \in A$, $\frac{zf'(z)}{f(z)} \neq 0$, $z \in E$, satisfies the differential subordination

$$(a - b)\frac{zf''(z)}{f(z)} + b\left(1 + \frac{zf''(z)}{f'(z)}\right) \prec ae^z + bz,$$

then

$$\frac{zf'(z)}{f(z)} \prec e^z, \ z \in E,$$

and hence $f(z)$ is starlike.

on choosing $a = 1$ and $b = \alpha$ in above corollary, we obtain:

Corollary 16. Let $\alpha$ be a non-zero real number such that $\Re(e^z) > -\alpha$. If $f \in A$, $\frac{zf'(z)}{f(z)} \neq 0$, $z \in E$, satisfies

$$(1 - \alpha)\frac{zf''(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right) \prec e^z + \alpha z,$$

Then, $f \in S^*.$

For $\phi(z) = g(z) = z$ in Theorem 13, we obtain the following result:

Corollary 17. Let $a$ and $b$ be non-zero real numbers such that $\Re(e^z) > -\frac{b}{a}$. If $f \in A$, $f'(z) \neq 0$, $z \in E$, satisfies

$$af'(z) + b\frac{zf''(z)}{f'(z)} \prec ae^z + bz,$$

then

$$f'(z) \prec e^z, \ z \in E,$$

and hence $f(z)$ is close-to-convex.

Remark 3. By selecting the dominant $q(z) = 1 + mz, 0 < m \leq 1$, we observed that the Condition 3 of Theorem 3 holds for all real numbers $a$ and $b(\neq 0)$ having same sign. Thus from Theorem 3, we have the following result.
Theorem 18. Let \( \phi \) be analytic function in the domain containing \( g(\mathbb{E}) \), where \( \phi(0) = 0 = \phi'(z) - 1 \) and \( \phi(w) \neq 0 \) for \( w \in g(\mathbb{E}) \setminus \{0\} \). Let real numbers \( a \) and \( b \neq 0 \) be such that \( \frac{a}{b} > 0 \). If \( f, g \in A \), \( \frac{zf'(z)}{\phi(g(z))} \neq 0 \), \( z \in \mathbb{E} \), satisfy

\[
a \frac{zf'(z)}{\phi(g(z))} + b \left( 1 + \frac{zf''(z)}{f'(z)} - \frac{z(\phi(g(z)))'}{\phi(g(z))} \right) < a(1 + mz) + \frac{bmz}{1 + mz},
\]

then

\[
\frac{zf'(z)}{\phi(g(z))} < 1 + mz, \text{ where } 0 < m \leq 1, \ z \in \mathbb{E}.
\]

Taking \( \phi(z) = z \) in above theorem, we get the following result:

Corollary 19. Let \( a \) and \( b \) are non-zero real numbers having same sign and \( 0 < m \leq 1 \). If \( f, g \in A \), \( \frac{zf'(z)}{g(z)} \neq 0 \), \( z \in \mathbb{E} \), satisfy

\[
a \frac{zf'(z)}{g(z)} + b \left( 1 + \frac{zf''(z)}{f'(z)} - \frac{zg'(z)}{g(z)} \right) < a(1 + mz) + \frac{bmz}{1 + mz},
\]

then

\[
\frac{zf'(z)}{g(z)} < 1 + mz, \ z \in \mathbb{E}.
\]

From Theorem 18, for \( \phi(z) = z \) and \( g(z) = f(z) \), we obtain:

Corollary 20. Let \( a \) and \( b \) be non-zero real numbers having same sign and \( 0 < m \leq 1 \). If \( f \in A \), \( \frac{zf'(z)}{f(z)} \neq 0 \), \( z \in \mathbb{E} \), satisfies

\[
(a - b) \frac{zf'(z)}{f(z)} + b \left( 1 + \frac{zf''(z)}{f'(z)} \right) < a(1 + mz) + \frac{bmz}{1 + mz},
\]

then

\[
\frac{zf'(z)}{f(z)} < 1 + mz, \ z \in \mathbb{E},
\]

and hence \( f(z) \) is starlike.

On selecting \( a = 1 \) and \( b = \alpha \) in above corollary, we get the following result:

Corollary 21. For \( \alpha > 0 \), if \( f \in A \), \( \frac{zf'(z)}{f(z)} \neq 0 \), \( z \in \mathbb{E} \), satisfies the differential subordination

\[
(1 - \alpha) \frac{zf'(z)}{f(z)} + \alpha \left( 1 + \frac{zf''(z)}{f'(z)} \right) < (1 + mz) + \frac{\alpha mz}{1 + mz},
\]

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then
\[
\frac{zf'(z)}{f(z)} < 1 + mz, \quad 0 < m \leq 1,
\]
and hence \( f(z) \) is starlike.

Selecting \( \phi(z) = g(z) = z \), in Theorem 18, we have:

**Corollary 22.** Let \( a \) and \( b (\neq 0) \) be real numbers having same sign. If \( f \in \mathcal{A} \), \( f'(z) \neq 0 \), \( z \in \mathbb{E} \), satisfies
\[
a f'(z) + b \frac{zf''(z)}{f'(z)} < a(1 + mz) + \frac{bmz}{1 + mz},
\]
then
\[
f'(z) < 1 + mz, \quad 0 < m \leq 1, \quad z \in \mathbb{E},
\]
and hence \( f(z) \) is close-to-convex.

**Remark 4.** Let \( q(z) = \frac{\beta(1 - z)}{\beta - z} \), then
\[
\Re \left( 1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} \right) = \Re \left( \frac{\beta - z^2}{(\beta - z)(1 - z)} \right) > 0, \quad \text{for } \beta > 1
\]
and
\[
\Re q(z) = \Re \left( \frac{\beta(1 - z)}{\beta - z} \right) > 0.
\]
In view of the above calculations, the Condition 3 of Theorem 3 is satisfied for real numbers \( a \) and \( b (\neq 0) \) such that \( \frac{a}{b} > 0 \). Consequently, we obtain the following result:

**Theorem 23.** Let \( \phi \) be analytic function in the domain containing \( g(\mathbb{E}) \), where \( \phi(0) = 0 = \phi'(z) - 1 \) and \( \phi(w) \neq 0 \) for \( w \in g(\mathbb{E}) \setminus \{0\} \). If \( f \in \mathcal{A} \), \( \frac{zf'(z)}{\phi(g(z))} \neq 0 \), \( z \in \mathbb{E} \), for real numbers \( a \), and \( b (\neq 0) \) such that \( \frac{a}{b} > 0 \), satisfies
\[
a \frac{zf'(z)}{\phi(g(z))} + b \left( 1 + \frac{zf''(z)}{f'(z)} - \frac{z(\phi(g(z))')'}{\phi(g(z))} \right) < a \frac{\beta(1 - z)}{\beta - z} + b \frac{(1 - \beta)z}{(\beta - z)(1 - z)},
\]
then
\[
\frac{zf'(z)}{\phi(g(z))} < \frac{\beta(1 - z)}{\beta - z}, \quad z \in \mathbb{E}, \quad \text{where } \beta > 1.
\]
Taking \( \phi(z) = z \), we get the following result from above theorem:
Corollary 24. If \( f, g \in A \), \( \frac{zf'(z)}{g(z)} \neq 0 \), \( z \in \mathbb{E} \), satisfy the differential subordination
\[
a \frac{zf'(z)}{g(z)} + b \left( 1 + \frac{zf''(z)}{f'(z)} - \frac{zg'(z)}{g(z)} \right) < \frac{a\beta(1-z)}{\beta - z} + \frac{b(1-\beta)z}{(\beta - z)(1-z)},
\]
then
\[
\frac{zf'(z)}{g(z)} < \frac{\beta(1-z)}{\beta - z}, \quad z \in \mathbb{E},
\]
where \( \beta > 1 \) and \( a, b \neq 0 \) are real numbers having same sign.

On selecting \( \phi(z) = z \) and \( g(z) = f(z) \) in Theorem 23, we obtain:

Corollary 25. Let \( a \) and \( b \neq 0 \) be real numbers having same sign and \( \beta > 1 \). If \( f \in A \), \( \frac{zf'(z)}{f(z)} \neq 0 \), \( z \in \mathbb{E} \), satisfies
\[
(1-a) \frac{zf'(z)}{f(z)} + b \left( 1 + \frac{zf''(z)}{f'(z)} \right) < \frac{a\beta(1-z)}{\beta - z} + \frac{b(1-\beta)z}{(\beta - z)(1-z)},
\]
then
\[
\frac{zf'(z)}{f(z)} < \frac{\beta(1-z)}{\beta - z}, \quad z \in \mathbb{E},
\]
and hence \( f(z) \) is starlike.

Choosing \( a = 1 \) and \( b = \alpha \) in above corollary, we get:

Corollary 26. For \( \alpha > 0 \), if \( f \in A \), \( \frac{zf'(z)}{f(z)} \neq 0 \), \( z \in \mathbb{E} \), satisfies the differential subordination
\[
(1-\alpha) \frac{zf'(z)}{f(z)} + \alpha \left( 1 + \frac{zf''(z)}{f'(z)} \right) < \frac{\beta(1-z)}{\beta - z} + \frac{\alpha(1-\beta)z}{(\beta - z)(1-z)},
\]
then
\[
\frac{zf'(z)}{f(z)} < \frac{\beta(1-z)}{\beta - z}, \quad \alpha > 1, \quad z \in \mathbb{E},
\]
i.e. \( f \in S^* \).

Taking \( \phi(z) = g(z) = z \) in Theorem 23, we have:
Corollary 27. Let \( a, b (\neq 0) \) be real numbers having same sign and \( \beta > 1 \). If \( f \in A, f'(z) \neq 0, z \in E, \) satisfies
\[
af'(z) + b \frac{zf''(z)}{f(z)} < \frac{a\beta(1-z)}{\beta-z} + \frac{b(1-\beta)z}{(\beta-z)(1-z)},
\]
then
\[
f'(z) < \frac{\beta(1-z)}{\beta-z}, \quad z \in E,
\]
and hence \( f(z) \) is close-to-convex.

Remark 5. On selecting the dominant \( q(z) = 1 + \frac{2}{3}z^2 \) in Theorem 3, it is easy to check that this dominant satisfies the Condition 3 of Theorem 3 for real numbers \( a \) and \( b \) of same sign, as
\[
\Re \left( 1 + \frac{zf''(z)}{f'(z)} - \frac{z(\phi(g(z)))'}{\phi(g(z))} \right) = 2\Re \left( 1 + \frac{2}{3}z^2 \right)^{-1} > 0
\]
and
\[
\Re q(z) = \Re \left( 1 + \frac{2}{3}z^2 \right) > 0.
\]
Consequently, we obtain the following result:

Theorem 28. For real numbers \( a \) and \( b (\neq 0) \) of same sign, if \( f, g \in A, \frac{zf'(z)}{\phi(g(z))} \neq 0, z \in E, \) satisfy
\[
a \frac{zf'(z)}{\phi(g(z))} + b \left( 1 + \frac{zf''(z)}{f'(z)} - \frac{z(\phi(g(z)))'}{\phi(g(z))} \right) < a \left( 1 + \frac{2}{3}z^2 \right) + \frac{4bz^2}{3 + 2z^2},
\]
then
\[
\frac{zf'(z)}{\phi(g(z))} < 1 + \frac{2}{3}z^2, \quad z \in E.
\]
Here, \( \phi \) is an analytic function in the domain containing \( g(E) \), such that \( \phi(0) = 0 = \phi'(z) - 1 \) and \( \phi(w) \neq 0 \) for \( w \in g(E) \setminus \{0\} \).

By selecting \( \phi(z) = z \) in above theorem, we obtain:

Corollary 29. Let \( a \) and \( b (\neq 0) \) be real numbers such that \( a > b \). If \( f, g \in A, \frac{zf'(z)}{g(z)} \neq 0, z \in E, \) satisfy
\[
a \frac{zf'(z)}{g(z)} + b \left( 1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{g(z)} \right) < a \left( 1 + \frac{2}{3}z^2 \right) + \frac{4bz^2}{3 + 2z^2},
\]
then
\[ \frac{zf'(z)}{g(z)} < 1 + \frac{2}{3}, \quad z \in \mathbb{E}. \]

On taking \( \phi(z) = z \) and \( g(z) = f(z) \) in Theorem 28, we have:

**Corollary 30.** Let \( a \) and \( b(\neq 0) \) be real numbers such that \( \frac{a}{b} > 0 \). If \( f \in \mathcal{A}, \ \frac{zf'(z)}{f(z)} \neq 0, \ z \in \mathbb{E}, \) satisfies
\[
(a - b)\frac{zf'(z)}{f(z)} + b \left(1 + \frac{zf''(z)}{f'(z)}\right) < a \left(1 + \frac{2}{3}z^2\right) + \frac{4bz^2}{3 + 2z^2},
\]
then
\[ \frac{zf'(z)}{f(z)} < 1 + \frac{2}{3}z^2, \quad z \in \mathbb{E}, \]
and hence \( f(z) \) is starlike.

If we take \( a = 1 \) and \( b = \alpha \) in above corollary, we get:

**Corollary 31.** For \( \alpha > 0 \), if \( f \in \mathcal{A}, \ \frac{zf'(z)}{f(z)} \neq 0, \ z \in \mathbb{E}, \) satisfies the differential subordination
\[
(1 - \alpha)\frac{zf'(z)}{f(z)} + \alpha \left(1 + \frac{zf''(z)}{f'(z)}\right) < \left(1 + \frac{2}{3}z^2\right) + \frac{4\alpha z^2}{3 + 2z^2},
\]
then
\[ \frac{zf'(z)}{f(z)} < 1 + \frac{2}{3}z^2, \quad z \in \mathbb{E}, \]
and hence \( f(z) \in \mathcal{S}^* \).

In Theorem 28, by selecting \( \phi(z) = g(z) = z \), we obtain:

**Corollary 32.** Let real numbers \( a \) and \( b(\neq 0) \) be such that \( \frac{a}{b} > 0 \). If \( f \in \mathcal{A}, \ f'(z) \neq 0, \ z \in \mathbb{E}, \) satisfies
\[
a f'(z) + b \frac{zf''(z)}{f'(z)} < a \left(1 + \frac{2}{3}z^2\right) + \frac{4bz^2}{3 + 2z^2},
\]
then
\[ f'(z) < 1 + \frac{2}{3}z^2, \quad z \in \mathbb{E}, \]
and hence \( f(z) \) is close-to-convex.
4. Sandwich Type Results

**Theorem 33.** Let \( a \) and \( b \neq 0 \) be real numbers such that \( \frac{a}{b} > 0 \). Let \( q, q(z) \neq 0 \) be univalent function in the unit disk \( \mathbb{E} \), with \( q(0) = 1 \) such that \( \frac{zq'(z)}{q(z)} \) is starlike univalent in \( \mathbb{E} \) and \( \Re q(z) > 0 \). Let \( \phi \) be analytic function in the domain containing \( g(\mathbb{E}) \), where \( \phi(0) = 0 = \phi'(0) - 1 \) and \( \phi(w) \neq 0 \) for \( w \in g(\mathbb{E}) \setminus \{0\} \). If \( f, g \in A \), \( \frac{zf'(z)}{\phi(g(z))} \in \mathcal{H}[g(0), 1] \cap Q \) with \( \frac{zf'(z)}{\phi(g(z))} \left( 1 + \frac{zf''(z)}{f'(z)} - \frac{z(\phi(g(z))'f'(z) - \phi(g(z))'}{\phi(g(z))} \right) \) is univalent in \( \mathbb{E} \), satisfy

\[
aq(z) + b \frac{zq'(z)}{q(z)} \prec a \frac{zf'(z)}{\phi(g(z))} + b \left( 1 + \frac{zf''(z)}{f'(z)} - \frac{z(\phi(g(z))'f'(z) - \phi(g(z))'}{\phi(g(z))} \right),
\]

then

\[
q(z) \prec \frac{zf'(z)}{\phi(g(z))}, \quad z \in \mathbb{E},
\]

and \( q(z) \) is the best subordinant.

**Proof:** Write \( p(z) = \frac{zf'(z)}{\phi(g(z))} \), then (5) becomes

\[
aq(z) + b \frac{zq'(z)}{q(z)} \prec ap(z) + b \frac{zp'(z)}{p(z)}.
\]

By defining \( \theta \) and \( \varphi \) as \( \theta(w) = aw \) and \( \varphi(w) = \frac{b}{w} \), where \( \theta \) and \( \varphi \) are analytic in \( \mathbb{C} \setminus \{0\} \) and \( \varphi(w) \neq 0, \ w \in \mathbb{C} \setminus \{0\} \). Therefore,

\[
Q_1(z) = zq'(z)\varphi(q(z)) = b \frac{zq'(z)}{q(z)}.
\]

A little calculation yields

\[
\frac{zQ_1(z)}{Q_1(z)} = 1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)}
\]

and

\[
\frac{\theta'(q(z))}{\varphi(q(z))} = \frac{aq(z)}{b}.
\]

In view of the given conditions, \( Q_1(z) \) is starlike and \( \Re \left[ \frac{\theta'(q(z))}{\varphi(q(z))} \right] > 0, \ z \in \mathbb{E} \). Hence the proof, now, follows from Lemma 2.
Theorem 34. Let \( q_1(z) \neq 0 \) and \( q_2(z) \neq 0 \) be univalent in \( E \) such that \( q_1(z) \) satisfies the condition of Theorem 33 whereas \( q_2(z) \) satisfies the Condition 3 of Theorem 3. Let \( \phi(z) \) be analytic function in the domain containing \( g(E) \) such that \( \phi(0) = 0 = \phi'(0) - 1 \) and \( \phi(w) \neq 0 \) for \( w \in g(E) \setminus \{0\} \). Let \( f, g \in A \), \( \frac{zf'(z)}{\phi(g(z))} \in H[1,1] \cap Q \) and
\[
a \frac{zf'(z)}{\phi(g(z))} + b \left( 1 + \frac{zf''(z)}{f'(z)} - \frac{z(\phi(g(z)))'}{\phi(g(z))} \right) \]be univalent in \( E \), where \( a \) and \( b(\neq 0) \) are real numbers. Further, if
\[
aq_1(z) + b \frac{zq_1'(z)}{q_1(z)} < a \frac{zf'(z)}{\phi(g(z))} + b \left( 1 + \frac{zf''(z)}{f'(z)} - \frac{z(\phi(g(z)))'}{\phi(g(z))} \right) < aq_2(z) + b \frac{zq_2'(z)}{q_2(z)},
\]
then
\[
q_1(z) < \frac{zf'(z)}{\phi(g(z))} < q_2(z), \ z \in E.
\]
Moreover, \( q_1(z) \) and \( q_2(z) \) are the best subordinant and the best dominant respectively.

Taking \( q_1(z) = 1 + mz \) and \( q_2(z) = 1 + nz \), \( 0 < m < n \leq 1 \), in Theorem 33, we have the following result:

Corollary 35. Let \( \phi(z) \) be an analytic function in the domain containing \( g \in E \) such that \( \phi(0) = 0 = \phi'(0) - 1 \) and \( \phi(w) \neq 0 \) for \( w \in g(E) \setminus \{0\} \). Let \( a, b(\neq 0) \) be real numbers such that \( \frac{a}{b} > 0 \). If \( f, g \in A \) be such that \( \frac{zf'(z)}{\phi(g(z))} \in H[1,1] \cap Q \) with
\[
a \frac{zf'(z)}{\phi(g(z))} + b \left( 1 + \frac{zf''(z)}{f'(z)} - \frac{z(\phi(g(z)))'}{\phi(g(z))} \right)
is univalent in \( E \) and satisfy
\[
a(1+mz) + b \frac{bmz}{1+sz} < a \frac{zf'(z)}{\phi(g(z))} + b \left( 1 + \frac{zf''(z)}{f'(z)} - \frac{z(\phi(g(z)))'}{\phi(g(z))} \right) < a(1+nz) + b \frac{zmz}{1+sz}
\]
then
\[
1 + mz < \frac{zf'(z)}{\phi(g(z))} < 1 + nz, \ z \in E,
\]
where \( m \) and \( n \) are real numbers, such that \( 0 < m < n \leq 1 \).

On selecting \( m = 1/4 \), \( n = 1/2 \) and \( a = 1 = b \) in above corollary, we obtain:

Example 1. Let \( \phi(z) \) be a analytic function in the domain containing \( g(E) \), where \( \phi(0) = 0 = \phi'(0) - 1 \) and \( \phi(w) \neq 0 \) for \( w \in g(E) \setminus \{0\} \). Let \( f, g \in A \) be such that \( \frac{zf'(z)}{\phi(g(z))} \in H[1,1] \cap Q \) with \( 1 + \frac{zf'(z)}{\phi(g(z))} + \frac{zf''(z)}{f'(z)} - \frac{z(\phi(g(z)))'}{\phi(g(z))} \) is univalent in \( E \), and satisfy
\[
\frac{z}{4} + \frac{z}{4-z} < \frac{zf'(z)}{\phi(g(z))} + \frac{zf''(z)}{f'(z)} - \frac{z(\phi(g(z)))'}{\phi(g(z))} < \frac{z}{2} + \frac{z}{2+z},
\]
(6)
then
\[
1 + \frac{z}{4} < \frac{zf'(z)}{\phi(g(z))} < 1 + \frac{z}{2}, \quad z \in \mathbb{E}.
\]  

(7)

In Example 1, on taking \(\phi(z) = z\), we get:

**Example 2.** Let \(f, g \in \mathcal{A}\) be such that 
\[
\frac{zf'(z)}{g(z)} \in \mathcal{H}[1,1] \cap \mathbb{Q} \text{ with } 1 + \frac{zf'(z)}{g(z)} + \frac{zf''(z)}{f'(z)} - \frac{zg'(z)}{g(z)} \text{ is univalent in } \mathbb{E} \text{ and satisfy}
\]
\[
\frac{z}{4} + \frac{z}{4+z} < \frac{zf'(z)}{g(z)} < \frac{z}{2} + \frac{z}{2+z}
\]

then
\[
1 + \frac{z}{4} < \frac{zf'(z)}{g(z)} < 1 + \frac{z}{2}, \quad z \in \mathbb{E}.
\]

On selecting \(\phi(z) = z\) and \(g(z) = f(z)\) in Example 1, we get:

**Example 3.** Suppose \(f \in \mathcal{A}\) is such that 
\[
\frac{zf'(z)}{f(z)} \in \mathcal{H}[1,1] \cap \mathbb{Q} \text{ with } 1 + \frac{zf''(z)}{f'(z)} \text{ is univalent in } \mathbb{E} \text{ and satisfies}
\]
\[
\frac{z}{4} + \frac{z}{4+z} < \frac{zf''(z)}{f'(z)} < \frac{z}{2} + \frac{z}{2+z}
\]

then
\[
1 + \frac{z}{4} < \frac{zf'(z)}{f(z)} < 1 + \frac{z}{2}, \quad z \in \mathbb{E}.
\]

On taking \(\phi(z) = g(z) = z\) in Example 1, we have:

**Example 4.** Suppose \(f \in \mathcal{A}\) is such that \(f'(z) \in \mathcal{H}[1,1] \cap \mathbb{Q}\) with \(f'(z) + \frac{zf''(z)}{f'(z)}\) is univalent in \(\mathbb{E}\) and satisfies
\[
1 + \frac{z}{4} + \frac{z}{4+z} < f'(z) + \frac{zf''(z)}{f'(z)} < 1 + \frac{z}{2} + \frac{z}{2+z},
\]

then
\[
1 + \frac{z}{4} < f'(z) < 1 + \frac{z}{2}, \quad z \in \mathbb{E}.
\]
Using Mathematica 10.0, we plot the images of the unit disk under the functions $\frac{z}{4} + \frac{z}{4+z}$ and $\frac{z}{2} + \frac{z}{2+z}$ of (6) in Figure 1 and $1 + \frac{z}{4}$ and $1 + \frac{z}{2}$ of (7) in Figure 2. It follows that if \[
\frac{zf'(z)}{\phi(g(z))} + \frac{zf''(z)}{f'(z)\phi(g(z))} - \frac{z(\phi(g(z)))'}{\phi(g(z))}\]
takes values in the light shaded portion.
of Figure 1, then $\frac{zf'(z)}{\phi(g(z))}$ will take values in the light shaded portion of Figure 2. Consequently, in view of Example 3 and Example 4, $f$ is starlike and close to convex respectively.

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