COEFFICIENT ESTIMATES FOR NEW SUBCLASSES OF
MEROMORPHICALLY BI-UNIVALENT FUNCTIONS

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ABSTRACT. In this paper, we have introduced and investigated three interesting subclasses \( \Sigma_{\lambda}^{*}(\alpha, \beta) \), \( \Sigma_{\lambda}^{*}(\beta, \gamma, \delta) \) and \( \tilde{\Sigma}_{\lambda}^{*}(\alpha, \beta, \gamma) \) of meromorphically bi-univalent functions defined on \( \Delta = \{ z \in \mathbb{C} : |z| > 1 \} \) and established their initial coefficient estimates.

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1. Introduction

Let \( A \) be the class of functions of the form :

\[
f(z) = z + \sum_{n=2}^{\infty} a_n z^n
\]  (1.1)

which are analytic in the unit disc \( U = \{ z \in \mathbb{C} : |z| < 1 \} \). Let \( S \) denote the subclass of \( A \), which consists of functions of the form (1.1) which are univalent and normalized by the conditions \( f(0) = 0 \) and \( f'(0) = 1 \) in \( U \).

A function \( f \in S \) is said to be starlike of order \( \alpha \) \((0 \leq \alpha < 1)\) in \( U \) if and only if

\[
\Re \left( \frac{zf'(z)}{f(z)} \right) > \alpha
\]

and is convex of order \( \alpha \) \((0 \leq \alpha < 1)\) in \( U \) if and only if

\[
\Re \left( 1 + \frac{zf''(z)}{f'(z)} \right) > \alpha .
\]

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We denote these subclasses respectively by \( S^\ast(\alpha) \) and \( K(\alpha) \).

Also, a function \( f \in S \) is said to be \( \delta \)-spirallike of order \( \gamma \) \((0 \leq \gamma < 1)\) in \( U \) if
\[
\Re \left( e^{i\delta} \frac{zf'(z)}{f(z)} \right) > \gamma \cos \delta,
\]
for some real \( \delta \) such that \(|\delta| < \frac{\pi}{2}\). The class of such functions is denoted by \( S^\gamma(\delta) \).

It is well known that every function \( f \in S \) has an inverse \( f^{-1} \), satisfying
\[
f^{-1}(f(z)) = z, \quad (z \in U) \text{ and } f^{-1}(f^{-1}(w)) = w, \quad (|w| < r_0(f); \quad r_0(f) \geq \frac{1}{4}).
\]

Let \( \Sigma \) denote the class of bi-univalent functions in \( U \) given by (1.1).

A systematic study of the class \( \Sigma \) was introduced in 1967 by Lewin [8], was revived in recent years by Srivastava et al. [10]. Ever since then, several authors investigated various subclasses of the class \( \Sigma \) and obtain estimates on the initial Taylor-Maclaurin coefficients \( |a_2| \) and \( |a_3| \) for functions in these subclasses. (see, for example, [[3], [12], [13]].)

In our present investigation, the concept of bi-univalency is extended to the class of meromorphic functions defined on \( \Delta = \{ z \in \mathbb{C} : |z| > 1 \} \).

The class of functions
\[
g(z) = z + b_0 + \sum_{n=1}^{\infty} \frac{b_n}{z^n} \quad (1.2)
\]
which are meromorphic and univalent in \( \Delta \) and is denoted by \( \Sigma^* \).

Since \( g \in \Sigma^* \) is univalent, it has an inverse \( g^{-1} = h \) that satisfies the following conditions:
\[
g^{-1}(g(z)) = z, \quad (z \in \Delta) \text{ and } g(g^{-1}(w)) = w, \quad (0 < M < |w| < \infty),
\]
where
\[
g^{-1}(w) = h(w) = w + B_0 + \sum_{n=1}^{\infty} \frac{B_n}{w^n} \quad (0 < M < |w| < \infty). \quad (1.3)
\]

A simple computation shows that
\[
w = g(h(w)) = (b_0 + B_0) + w + \frac{b_1 + B_1}{w} + \frac{B_2 - b_1 B_0 + b_2}{w^2} + \frac{B_3 - b_1 B_1 + b_1 B_0^2 - 2b_2 B_0 + b_3 + \ldots}{w^3} \quad (1.4)
\]
Comparing with initial coefficients in (1.4), we find that
\[
\begin{align*}
b_0 + B_0 &= 0 \quad \Rightarrow \quad B_0 = -b_0 \\
b_1 + B_1 &= 0 \quad \Rightarrow \quad B_1 = -b_1 \\
B_2 - b_1 B_0 + b_2 &= 0 \quad \Rightarrow \quad B_2 = -(b_2 + b_0 b_1)
\end{align*}
\]

\[ B_3 - b_1 B_1 + b_1 B_0^2 - 2 b_2 B_0 + b_3 = 0 \implies B_3 = -(b_3 + 2b_0b_2 + b_0^2b_1 + b_1^2). \]

Equation (1.3) becomes,

\[ g^{-1}(w) = h(w) = w - b_0 - \frac{b_2 + b_0b_1}{w} - \frac{b_3 + 2b_0b_2 + b_0^2b_1 + b_1^2}{w^3} - \ldots \quad (1.5) \]

Analogues to the bi-univalent analytic functions, a function \( g \in \Sigma^* \) is said to be meromorphic bi-univalent function if \( g^{-1} \in \Sigma^* \). The class of all meromorphic bi-univalent functions is denoted by \( \Sigma^*_B \).

Estimates on the coefficients of meromorphically bi-univalent functions were widely investigated in the literature of Geometric function theory. Recently several researchers such as Halim et al.\[5\], Hamidi et al.\[6, 7\], Srivastava et al.\[9\] and Xiao et al.\[11\], introduced new subclasses of meromorphic bi-univalent functions and obtained estimates on the initial coefficients \( |b_0| \) and \( |b_1| \). Also in \[1\], Babalola defined and studied the class \( \ell_{\lambda}(\beta) \) of \( \lambda \)-pseudo starlike functions of order \( \beta \).

Motivated by the aforementioned work, in our present investigation, we introduce three new subclasses of the class \( \Sigma^*_B \) and obtained the estimates on the initial coefficients.

In order to derive our main results, we recall here the following Lemma.

**Lemma 1.** (\[4\], see also \([2, \text{p.41}]\)). Let \( p \in P \), where \( P \) is the family of all functions \( p \), analytic in \( \Delta \) for which \( \Re \{p(z)\} > 0 \) and have the form

\[ p(z) = 1 + \frac{p_1}{z} + \frac{p_2}{z^2} + \frac{p_3}{z^3} + \ldots, (z \in \Delta). \]

Then \( |p_n| \leq 2 \) for each \( n \in \mathbb{N} \).

**2. COEFFICIENT BOUNDS FOR THE FUNCTION CLASS \( \Sigma^*_B,\lambda(\alpha, \beta) \)**

We define the class \( \Sigma^*_B,\lambda(\alpha, \beta) \) as follows:

**Definition 1.** A function \( g \in \Sigma^*_B \) given by (1.2) is said to be in the class \( \Sigma^*_B,\lambda(\alpha, \beta) \) if the following conditions are satisfied:

\[ \left| \arg \left( \frac{z[g'(z)]^\lambda}{(1-\beta)g(z) + \beta z g'(z)} \right) \right| < \frac{\alpha \pi}{2} \quad (z \in \Delta) \quad (2.1) \]

and

\[ \left| \arg \left( \frac{w[h'(w)]^\lambda}{(1-\beta)h(w) + \beta wh'(w)} \right) \right| < \frac{\alpha \pi}{2} \quad (w \in \Delta), \quad (2.2) \]

where \( 0 < \alpha \leq 1, 0 \leq \beta < 1, \lambda \geq 1 \) and the function \( h \) is the inverse of \( g \) given by (1.5).
We denote by $\Sigma_{B,\lambda}^*(\alpha, \beta)$, the class of functions which are meromorphic strongly $\lambda$-pseudo starlike bi-univalent of order $\alpha$ in $\Delta$.

The estimates on the coefficients $|b_0|$ and $|b_1|$ for the class $\Sigma_{B,\lambda}^*(\alpha, \beta)$ are given as below.

**Theorem 1.** Let $g$ given by (1.2) be in the class $\Sigma_{B,\lambda}^*(\alpha, \beta)$. Then

$$|b_0| \leq \frac{2\alpha}{1-\beta}$$

and

$$|b_1| \leq \frac{2\sqrt{5} \alpha^2}{1-2\beta + \lambda}.$$  \hspace{1cm} (2.4)

**Proof.** Let $g \in \Sigma_{B,\lambda}^*(\alpha, \beta)$. Then by Definition 1, the conditions (2.1) and (2.2) can be rewritten as

$$\frac{z[g'(z)]^\lambda}{(1-\beta)g(z) + \beta zg'(z)} = [p(z)]^\alpha$$  \hspace{1cm} (2.5)

and

$$\frac{w[h'(w)]^\lambda}{(1-\beta)h(w) + \beta wh'(w)} = [q(w)]^\alpha$$  \hspace{1cm} (2.6)

respectively. Where $p(z), q(w) \in P$ and have the forms

$$p(z) = 1 + \frac{p_1}{z} + \frac{p_2}{z^2} + \frac{p_3}{z^3} + \ldots \ (z \in \Delta)$$

and

$$q(w) = 1 + \frac{q_1}{w} + \frac{q_2}{w^2} + \frac{q_3}{w^3} + \ldots \ (w \in \Delta).$$

Clearly,

$$[p(z)]^\alpha = 1 + \frac{\alpha p_1}{z} + \frac{\alpha p_2}{z^2} + \frac{\alpha(\alpha-1)p_3}{z^3} + \frac{1}{6}\alpha(\alpha-1)(\alpha-2)p_1^3 + \alpha(\alpha-1)p_1p_2 + \alpha p_3$$

and

$$[q(w)]^\alpha = 1 + \frac{\alpha q_1}{w} + \frac{\alpha q_2}{w^2} + \frac{\alpha(\alpha-1)q_3}{w^3} + \frac{1}{6}\alpha(\alpha-1)(\alpha-2)q_1^3 + \alpha(\alpha-1)q_1q_2 + \alpha q_3$$

+ $\ldots$. 

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Also,

\[
\frac{z [g'(z)]^\lambda}{(1 - \beta) g(z) + \beta z g'(z)} = 1 - \frac{(1 - \beta)b_0}{z} + \frac{[(1 - \beta)^2 b_0^2 - (1 - 2\beta + \lambda)b_1]}{z^2} - \frac{[(1 - \beta)^3 b_0^3 - (1 - \beta)(2 - 4\beta + \lambda)b_0 b_1 + (1 - 3\beta + 2\lambda)b_2]}{z^3} + \ldots
\]

and

\[
\frac{w [h'(w)]^\lambda}{(1 - \beta) h(w) + \beta w h'(w)} = 1 + \frac{(1 - \beta)b_0}{w} + \frac{[(1 - \beta)^2 b_0^2 + (1 - 2\beta + \lambda)b_1]}{w^2} + \frac{[(1 - \beta)^3 b_0^3 + (1 + 2\lambda + (1 - \beta)(2 - 4\beta + \lambda))b_0 b_1 + (1 + 3\beta + 2\lambda)b_2]}{w^3} + \ldots
\]

Now, equating the coefficients in (2.5) and (2.6), we get

\[-(1 - \beta)b_0 = \alpha p_1, \quad (2.7)\]

\[(1 - \beta)^2 b_0^2 - (1 - 2\beta + \lambda)b_1 = \alpha p_2 + \frac{\alpha(\alpha - 1)}{2} p_1^2, \quad (2.8)\]

\[(1 - \beta)b_0 = \alpha q_1, \quad (2.9)\]

\[(1 - \beta)^2 b_0^2 + (1 - 2\beta + \lambda)b_1 = \alpha q_2 + \frac{\alpha(\alpha - 1)}{2} q_1^2. \quad (2.10)\]

From equations (2.7) and (2.9), we get

\[p_1 = -q_1 \quad (2.11)\]

and

\[2(1 - \beta)^2 b_0^2 = \alpha^2 (p_1^2 + q_1^2). \]

Using (2.11), we have

\[b_0^2 = \frac{\alpha^2 p_1^2}{(1 - \beta)^2}. \quad (2.12)\]

Applying Lemma 1, for the coefficient \(p_1\) we have

\[|b_0|^2 \leq \frac{4\alpha^2}{(1 - \beta)^2} \Rightarrow |b_0| \leq \frac{2\alpha}{1 - \beta}. \]

Which gives the bound on \(|b_0|\) as given in (2.3).

Next, in order to find the bound on \(|b_1|\), by using the equations (2.8) and (2.10), we
get

\[ (1 - \beta)^4 b_0^4 - (1 - 2\beta + \lambda)^2 b_1^2 = \frac{1}{4} \alpha^2 (\alpha - 1)^2 p_1^2 q_1^2 + \frac{1}{2} \alpha^2 (\alpha - 1) (p_1^2 q_2 + p_2 q_1^2) - \alpha^2 p_2 q_1. \]

By simplifying and using (2.12), we have

\[ (1 - 2\beta + \lambda)^2 b_1^2 = \alpha^4 p_1 - \frac{1}{4} \alpha^2 (\alpha - 1)^2 p_1^2 q_1^2 - \frac{1}{2} \alpha^2 (\alpha - 1) (p_1^2 q_2 + p_2 q_1^2) - \alpha^2 p_2 q_1. \]

Applying Lemma 1, for the coefficients \( p_1, q_1, p_2 \) and \( q_2 \) we get

\[ |b_1|^2 \leq 16 \alpha^4 + 4 \alpha^2 (\alpha - 1)^2 + 8 \alpha^2 (\alpha - 1) + 4 \alpha^2. \]

\[ |b_1|^2 \leq \frac{20 \alpha^4}{(1 - 2\beta + \lambda)^2}, \]

\[ \Rightarrow |b_1| \leq \frac{2 \sqrt{5} \alpha^2}{1 - 2\beta + \lambda}. \]

Which gives the bound on \(|b_1|\) as given in (2.4).

This completes the proof of Theorem 1.

3. Coefficient bounds for the function class \( \Sigma^*_B(\lambda, \beta, \gamma) \)

The definition of the class \( \Sigma^*_B(\lambda, \beta, \gamma) \) is as follows:

**Definition 2.** A function \( g \in \Sigma^*_B \) given by (1.2) is said to be in the class \( \Sigma^*_B(\lambda, \beta, \gamma) \) if the following conditions are satisfied:

\[ \Re \left( \frac{z g'(z)^\lambda}{(1 - \beta) g(z) + \beta z g'(z)} \right) > \gamma \quad (z \in \Delta) \quad (3.1) \]

and

\[ \Re \left( \frac{w h'(w)^\lambda}{(1 - \beta) h(w) + \beta w h'(w)} \right) > \gamma \quad (w \in \Delta), \quad (3.2) \]

where \( 0 \leq \beta, \gamma < 1, \lambda \geq 1 \) and the function \( h \) is the inverse of \( g \) given by (1.5).

We denote \( \Sigma^*_B(\lambda, \beta, \gamma) \) the class of meromorphically \( \lambda \)-pseudo starlike bi-univalent function of order \( \gamma \).

We now derive the estimates on the coefficients \(|b_0|\) and \(|b_1|\) for the meromorphically bi-univalent function class \( \Sigma^*_B(\lambda, \beta, \gamma) \).
Theorem 2. Let \( g \) given by (1.2) be in the class \( \Sigma_B^*(\lambda, \beta, \gamma) \). Then

\[
|b_0| \leq \frac{2(1 - \gamma)}{1 - \beta}
\]  
(3.3)

and

\[
|b_1| \leq \frac{2(1 - \gamma)\sqrt{4\gamma^2 - 8\gamma + 5}}{1 - 2\beta + \lambda}.
\]  
(3.4)

Proof. Let \( g \in \Sigma_B^*(\lambda, \beta, \gamma) \). Then by Definition 2, the conditions (3.1) and (3.2) can be rewritten as follows:

\[
\frac{z[g'(z)]^\lambda}{(1 - \beta)g(z) + \beta z g'(z)} = \gamma + (1 - \gamma)p(z)
\]  
(3.5)

and

\[
\frac{w[h'(w)]^\lambda}{(1 - \beta)h(w) + \beta w h'(w)} = \gamma + (1 - \gamma)q(w)
\]  
(3.6)

respectively. Where \( p(z), q(w) \in P \) and have the forms

\[
p(z) = 1 + \frac{p_1}{z} + \frac{p_2}{z^2} + \frac{p_3}{z^3} + \ldots \quad (z \in \Delta)
\]

and

\[
q(w) = 1 + \frac{q_1}{w} + \frac{q_2}{w^2} + \frac{q_3}{w^3} + \ldots \quad (w \in \Delta).
\]

Clearly,

\[
\gamma + (1 - \gamma)p(z) = 1 + \frac{(1 - \gamma)p_1}{z} + \frac{(1 - \gamma)p_2}{z^2} + \frac{(1 - \gamma)p_3}{z^3} + \ldots
\]

and

\[
\gamma + (1 - \gamma)q(w) = 1 + \frac{(1 - \gamma)q_1}{w} + \frac{(1 - \gamma)q_2}{w^2} + \frac{(1 - \gamma)q_3}{w^3} + \ldots.
\]

Also,

\[
\frac{z[g'(z)]^\lambda}{(1 - \beta)g(z) + \beta z g'(z)} = 1 - \frac{(1 - \beta)b_0}{z} + \frac{[1 - \beta]b_0^2}{z^2} - \frac{1 - 2\beta + \lambda}{b_1} \frac{b_0}{z} + \frac{[(1 - \beta)^2b_0^2 - (1 - 2\beta + \lambda)b_1]}{z^2} - \frac{[(1 - \beta)^3b_0^3 - (1 - \beta)(2 - 4\beta + \lambda)b_0b_1 + (1 - 3\beta + 2\lambda)b_2]}{z^3} + \ldots
\]
and

\[
\begin{align*}
\frac{w(h'(w))^{\lambda}}{(1 - \beta)h(w) + \beta wh'(w)} &= 1 + \frac{(1 - \beta)b_0}{w} + \frac{[(1 - \beta)^2 b_0^2 + (1 - 2\beta + \lambda)b_1]}{w^2} \\
&\quad + \frac{[(1 - \beta)^3 b_0^3 + (1 + 2\lambda + (1 - \beta)(2 - 4\beta + \lambda))b_0 b_1 + (1 + 3\beta + 2\lambda)b_2]}{w^3} + \ldots.
\end{align*}
\]

Now, equating the coefficients in (3.5) and (3.6), we get

\[
\begin{align*}
-(1 - \beta)b_0 &= (1 - \gamma)p_1, \quad \text{(3.7)} \\
(1 - \beta)^2 b_0^2 - (1 - 2\beta + \lambda)b_1 &= (1 - \gamma)p_2, \quad \text{(3.8)} \\
(1 - \beta)b_0 &= (1 - \gamma)q_1, \quad \text{(3.9)} \\
(1 - \beta)^2 b_0^2 + (1 - 2\beta + \lambda)b_1 &= (1 - \gamma)q_2. \quad \text{(3.10)}
\end{align*}
\]

From equations (3.7) and (3.9), we get

\[
p_1 = -q_1 \quad \text{(3.11)}
\]

and

\[
2(1 - \beta)^2 b_0^2 = (1 - \gamma)^2 (p_1^2 + q_1^2),
\]

Using (3.11), we have

\[
b_0^2 = \frac{(1 - \gamma)^2 p_1^2}{(1 - \beta)^2}. \quad \text{(3.12)}
\]

Applying Lemma 1 for the coefficient \(p_1\), we have

\[
|b_0|^2 \leq \frac{4(1 - \gamma)^2}{(1 - \beta)^2} \quad \Rightarrow \quad |b_0| \leq \frac{2(1 - \gamma)}{1 - \beta}. \quad \text{(3.13)}
\]

Which is the bound on \(|b_0|\) as given in (3.3).

Next, in order to find the bound on \(|b_1|\), by using the equations (3.8) and (3.10), we get

\[
(1 - \beta)^4 b_0^4 - (1 - 2\beta + \lambda)^2 b_1^2 = (1 - \gamma)^2 p_2 q_2.
\]

By simplifying and using (3.12), we have

\[
(1 - 2\beta + \lambda)^2 b_1^2 = (1 - \gamma)^4 p_1^4 - (1 - \gamma)^2 p_2 q_2.
\]
Applying Lemma 1 for the coefficients $p_1$, $p_2$ and $q_2$ we get
\[(1 - 2\beta + \lambda)^2 |b_1|^2 \leq 16(1 - \gamma)^4 + 4(1 - \gamma)^2.\]
\[|b_1|^2 \leq \frac{4(1 - \gamma)^2 [4\gamma^2 - 8\gamma + 5]}{(1 - 2\beta + \lambda)^2} , \]
\[\Rightarrow |b_1| \leq \frac{2(1 - \gamma) \sqrt{4\gamma^2 - 8\gamma + 5}}{1 - 2\beta + \lambda}.
\]

Which gives the bound on $|b_1|$ as given in (3.4). This completes the proof of Theorem 2.

4. Coefficient bounds for the function class $\tilde{\Sigma}^*_B,\lambda(\beta, \gamma, \delta)$

For the function $g$ given by (1.2) with $b_1 = b_2 = ... = b_{k-1} = 0$, some estimates on the initial coefficients can be obtained. We define the class $\tilde{\Sigma}^*_B,\lambda(\beta, \gamma, \delta)$ as follows:

**Definition 3.** A function
\[g(z) = z + b_0 + \sum_{n=k}^{\infty} \frac{b_n}{z^n} \tag{4.1}\]
is said to be in the class $\tilde{\Sigma}^*_B,\lambda(\beta, \gamma, \delta)$ where $0 \leq \beta, \gamma < 1$, $\lambda \geq 1$ and $|\delta| < \frac{\pi}{2}$, if the following conditions are satisfied:
\[\Re \left( \frac{e^{i\delta} z [g'(z)]^\lambda}{(1 - \beta)g(z) + \beta z g'(z)} \right) > \gamma \cos \delta \quad (z \in \Delta) \tag{4.2}\]
and
\[\Re \left( \frac{e^{i\delta} w [h'(w)]^\lambda}{(1 - \beta)h(w) + \beta w h'(w)} \right) > \gamma \cos \delta \quad (w \in \Delta), \tag{4.3}\]
where the function $h$ is the inverse of $g$ given by
\[h(w) = w - b_0 - \frac{b_k}{w^k} - \frac{kb_0 b_k + b_{k+1}}{w^{k+1}} - \ldots . \tag{4.4}\]

We call $\tilde{\Sigma}^*_B,\lambda(\beta, \gamma, \delta)$ the class of weakerly meromorphic $\lambda$-pseudo $\delta$-spiralike bi-univalent functions of order $\gamma$.

We now derive the estimates on the coefficients for the function class $\tilde{\Sigma}^*_B,\lambda(\beta, \gamma, \delta)$, we find the following result.
Theorem 3. Let \( g \) given by (4.1) be in the class \( \tilde{\Sigma}_{B,\lambda}^{*}(\beta, \gamma, \delta) \). Then

\[
|b_0| \leq \frac{1}{2k} \left[ \frac{4(1 + \gamma(\gamma - 2)\cos^2 \delta)}{1 - \beta} \right]^{1/2}
\]

(4.5)

and

(a) for each positive odd integer \( k \),

\[
|b_k| \leq \frac{2}{1 + \lambda k - \beta(1 + k)} \sqrt{\left[ 1 + \gamma(\gamma - 2)\cos^2 \delta \right] \left[ 1 + 4\frac{k}{k} (1 + \gamma(\gamma - 2)\cos^2 \delta) \right]}^{1/2}
\]

(4.6)

(b) for each positive even integer \( k \),

\[
|b_k| \leq \frac{2}{1 + \lambda k - \beta(1 + k)} \sqrt{\left[ 1 + \gamma(\gamma - 2)\cos^2 \delta \right] \left[ 1 + 2\frac{k}{k} (1 + \gamma(\gamma - 2)\cos^2 \delta) \right]}^{1/2}
\]

(4.7)

Proof. Let \( g(z) = z + b_0 + \sum_{n=k}^{\infty} b_n z^n \). Then by Definition 3, the conditions (4.2) and (4.3) can be rewritten as follows:

\[
e^{i\delta} z \left[ g'(z) \right]^\lambda = \gamma \cos \delta + (e^{i\delta} - \gamma \cos \delta) p(z)
\]

(4.8)

and

\[
e^{i\delta} w \left[ h'(w) \right]^\lambda = \gamma \cos \delta + (e^{i\delta} - \gamma \cos \delta) q(w)
\]

(4.9)

respectively. Where \( p(z), q(w) \in P \) and have the forms

\[
p(z) = 1 + \frac{p_1}{z} + \frac{p_2}{z^2} + \frac{p_3}{z^3} + \ldots \quad (z \in \Delta)
\]

and

\[
q(w) = 1 + \frac{q_1}{w} + \frac{q_2}{w^2} + \frac{q_3}{w^3} + \ldots \quad (w \in \Delta)
\]

Clearly,

\[
\gamma \cos \delta + (e^{i\delta} - \gamma \cos \delta)p(z) = e^{i\delta} + \frac{(e^{i\delta} - \gamma \cos \delta)p_1}{z} + \ldots + \frac{(e^{i\delta} - \gamma \cos \delta)p_k}{z^k} + \frac{(e^{i\delta} - \gamma \cos \delta)p_{k+1}}{z^{k+1}} + \ldots
\]
and
\[ \gamma \cos \delta + (e^{i \delta} - \gamma \cos \delta)q(w) = e^{i \delta} + \frac{(e^{i \delta} - \gamma \cos \delta)q_1}{w} + \ldots + \frac{(e^{i \delta} - \gamma \cos \delta)q_k}{w^k} + \frac{(e^{i \delta} - \gamma \cos \delta)q_{k+1}}{w^{k+1}} + \ldots \]

Also,
\[ e^{i \delta} z\left[ g'(z) \right]^\lambda = e^{i \delta} - e^{i \delta} \frac{(1 - \beta)b_0}{z} + \ldots + e^{i \delta} \frac{(-1)^k(1 - \beta)^k b_0}{z^k} + \ldots \]
\[ + e^{i \delta} \frac{((-1)^{k+1}(1 - \beta)^{k+1}b_0^{(k+1)} - (1 + \lambda k - \beta(1 + k))b_k}{z^{k+1}} + \ldots \]

and using equation (4.4), we get
\[ e^{i \delta} w\left[ h'(w) \right]^\lambda = e^{i \delta} + e^{i \delta} \frac{(1 - \beta)b_0}{w} + \ldots + e^{i \delta} \frac{(1 - \beta)^k b_0}{w^k} \]
\[ + e^{i \delta} \frac{((1 - \beta)^{k+1}b_0^{(k+1)} + (1 + \lambda k - \beta(1 + k))b_k}{w^{k+1}} + \ldots \]

Now, equating the coefficients in (4.8) and (4.9), we get
\[ e^{i \delta} (-1)^k(1 - \beta)^k b_0^k = (e^{i \delta} - \gamma \cos \delta)p_k \] (4.10)
\[ e^{i \delta} \left[ (-1)^{k+1}(1 - \beta)^{k+1}b_0^{(k+1)} - (1 + \lambda k - \beta(1 + k))b_k \right] = (e^{i \delta} - \gamma \cos \delta)p_{k+1} \] (4.11)
\[ e^{i \delta} (1 - \beta)^k b_0^k = (e^{i \delta} - \gamma \cos \delta)q_k \]
\[ e^{i \delta} \left[ (1 - \beta)^{k+1}b_0^{(k+1)} + (1 + \lambda k - \beta(1 + k))b_k \right] = (e^{i \delta} - \gamma \cos \delta)q_{k+1} \] (4.12)

From equations (4.10), we get
\[ b_0^k = \frac{(e^{i \delta} - \gamma \cos \delta)p_k}{e^{i \delta} (-1)^k(1 - \beta)^k} \]

Using Lemma 1, we get
\[ |b_0|^k \leq \frac{2|(e^{i \delta} - \gamma \cos \delta)|}{(1 - \beta)^k} \]
\[ |b_0| \leq \left[ \frac{4(1 + \gamma(\gamma - 2\cos^2 \delta))}{1 - \beta} \right]^{\frac{1}{2k}} \]
Which is the bound on $|b_0|$, as asserted in (4.5).

Next, in order to find the bound on $|b_k|$, for each positive odd integer $k$, multiplying both sides of (4.11) by both sides of (4.12), respectively we get

$$e^{2i\delta} \left[ (1 - \beta)^{2k+2} b_0^{2k+2} - (1 + \lambda k - \beta(1 + k))^2 b_k^2 \right] = (e^{i\delta} - \gamma \cos \delta)^2 p_{k+1} q_{k+1},$$

$$[1 + \lambda k - \beta(1 + k)]^2 b_k^2 = -\frac{(e^{i\delta} - \gamma \cos \delta)^2 p_{k+1} q_{k+1}}{e^{2i\delta}} + (1 - \beta)^{2k+2} b_0^{2k+2}.$$

By using Lemma 1 and considering the bound on $|b_0|$, we conclude that

$$|b_k| \leq 2 \sqrt{1 + \gamma(\gamma - 2) \cos^2 \delta} \left[ 1 + \frac{1}{1 + 4\hat{k}(1 + \gamma(\gamma - 2) \cos^2 \delta) \hat{k}} \right].$$

(4.13)

On the other hand, for every positive even integer $k$, from (4.12) and using the Lemma 1 and also considering the bound on $|b_0|$, we conclude that

$$|b_k| \leq 2 \sqrt{1 + \gamma(\gamma - 2) \cos^2 \delta} \left[ 1 + \frac{1}{1 + 2\hat{k}(1 + \gamma(\gamma - 2) \cos^2 \delta) 2\hat{k}} \right].$$

(4.14)

Equations (4.13) and (4.14) gives the bound on $|b_k|$ as asserted in (4.6) and (4.7) respectively. Hence, complete the proof of Theorem 3.

**Remark 1.** By suitably specializing the various parameters involved in the assertion of Theorem 1, Theorem 2 and Theorem 3, we can deduce the corresponding coefficient estimates for several simpler meromorphically bi-univalent function classes.

**References**


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