ON SUBCLASSES OF BI-UNIVALENT FUNCTIONS ASSOCIATED WITH THE RÁDUCAŅU-ORHAN DIFFERENTIAL OPERATOR

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Abstract. In this paper, we obtain estimates on the coefficients $|a_2|$ and $|a_3|$ for the functions of certain new subclasses of the bi-univalent function class $\Sigma$ defined on the open unit disk $U = \{ z \in \mathbb{C} : |z| < 1 \}$, which are associated with the Ráducanu-Orhan differential operator. Moreover, connections to the earlier known results are indicated.

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1. Introduction

Let $\mathcal{A} = \left\{ f : U \to \mathbb{C} : f \text{ is analytic in the unit disk } U, f(0) = 0, f'(0) = 1 \right\}$ be the class of functions of the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \quad (1)$$

and the subclass of $\mathcal{A}$ consisting the univalent functions in $U$ is denoted by $\mathcal{S}$. It is clear from the Koebe one quarter theorem (see [4]) that every function $f \in \mathcal{S}$ has an inverse $f^{-1}$, defined by

$$f^{-1}(f(z)) = z, \ (z \in U) \text{ and } f(f^{-1}(w)) = w, \ (|w| < r_0(f), r_0(f) \geq 1/4).$$

In fact, we have:

$$f^{-1}(w) = g(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \cdots, \quad (2)$$

where $g$ be an extension of $f^{-1}$ to $U$. A function $f \in \mathcal{S}$ is said to be bi-univalent in $U$ if both $f$ and $f^{-1}$ are univalent in $U$. Let $\Sigma$ denote the class of bi-univalent functions in $U$ given by (1).
For more details about the bi-univalent function class $\Sigma$, see Lewin [6], Netanyahu [7], Brannan and Clunie [2], Srivastava et al. [14] etc. Also Brannan and Taha [3], (see also [15]) introduced $\mathcal{S}_\alpha^\ast \Sigma$, the class of strongly bi-starlike functions of order $\alpha$ where $0 < \alpha \leq 1$ and $\mathcal{S}_\beta^\ast \Sigma$, the class of bi-starlike functions of order $\beta$ where $0 \leq \beta < 1$ and found the estimates on the coefficients $|a_2|$ and $|a_3|$ for the functions in these subclasses. In recent investigations many researchers (viz. [5, 10, 13] etc.) introduced various subclasses of the function class $\Sigma$ and obtained the non-sharp estimates on $|a_2|$ and $|a_3|$ for the functions in these subclasses.

For $f(z)$ given by (1) and $j(z) = z + \sum_{k=2}^{\infty} b_k z^k$, the Hadamard product or convolution is given by

$$(f * j)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k, \quad z \in \mathbb{U}.$$  

For $f \in A$ and $0 \leq \mu \leq \delta$, $n \in \mathbb{N} := \{1, 2, 3, \cdots\}$; Răducanu and Orhan [11] introduced the following differential operator:

$$D_{\delta \mu}^0 f(z) = f(z),$$

$$D_{\delta \mu}^1 f(z) = D_{\delta \mu} f(z) = \delta \mu z^2 f''(z) + (\delta - \mu) z f'(z) + (1 - \delta + \mu) f(z),$$

$$D_{\delta \mu}^n f(z) = D_{\delta \mu}^{n-1} f(z).$$  

See that, for the function $f$ given by (1), this becomes:

$$D_{\delta \mu}^n f(z) = z + \sum_{k=2}^{\infty} F_k(\delta, \mu, n) a_k z^k$$

or

$$D_{\delta \mu}^n f(z) = (f * j)(z),$$

where

$$j(z) = z + \sum_{k=2}^{\infty} F_k(\delta, \mu, n) z^k$$

and

$$F_k(\delta, \mu, n) = [1 + (\delta \mu k + \delta - \mu) (k - 1)]^n.$$

Observe that for $\mu = 0$ we get the Al-Oboudi differential operator (see [1]) and for $\mu = 0$, $\delta = 1$ we get the Sălăgean differential operator (see [12]).

The object of the present paper is to introduce the subclasses $\mathcal{B}_{\delta \mu}^n \Sigma (n, \alpha, \lambda)$ and $\mathcal{H}_{\delta \mu}^n \Sigma (n, \beta, \lambda)$ of the function class $\Sigma$, which are associated with the Răducanu-Orhan differential operator and to obtain estimates on $|a_2|$ and $|a_3|$ for the functions in these new subclasses using similar techniques used by Srivastava et al. [14].

We need the following lemma (see [9]) to prove our main results.
Lemma 1. If \( p(z) \in \mathcal{P} \), the Carathéodory class of analytic functions with positive real part in \( \mathbb{U} \), then \(|p_n| \leq 2\) for each \( n \in \mathbb{N} \), where

\[
p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots, \quad (z \in \mathbb{U}).
\]

2. Main Results

Definition 1. A function \( f(z) \) given by (1) is said to be in the class \( \mathcal{B}_{\Sigma}^{\delta \mu}(n, \alpha, \lambda) \) if the following conditions are satisfied:

\[
f \in \Sigma, \quad \left| \frac{\arg \left\{ \left(1 - \lambda\right) D_{\delta \mu}^n f(z) + \lambda D_{\delta \mu}^{n+1} f(z) \right\}}{z} \right| < \frac{\alpha \pi}{2}
\]

\((0 < \alpha \leq 1, 0 \leq \mu \leq \delta, \lambda \geq 1, n \in \mathbb{N}_0, z \in \mathbb{U})\)

and

\[
\left| \frac{\arg \left\{ \left(1 - \lambda\right) D_{\delta \mu}^n g(w) + \lambda D_{\delta \mu}^{n+1} g(w) \right\}}{w} \right| < \frac{\alpha \pi}{2}
\]

\((0 < \alpha \leq 1, 0 \leq \mu \leq \delta, \lambda \geq 1, n \in \mathbb{N}_0, w \in \mathbb{U})\),

where the function \( g \) is given by (2).

Theorem 2. If the function \( f(z) \) given by (1) be in the class \( \mathcal{B}_{\Sigma}^{\delta \mu}(n, \alpha, \lambda) \), then

\[
|a_2| \leq \frac{2\alpha}{\sqrt{2\alpha[1 + 2(3\delta \mu + \delta - \mu)]^n[1 + 2\lambda(3\delta \mu + \delta - \mu)]}} \quad (3)
\]

and

\[
|a_3| \leq \frac{4\alpha^2}{[1 + 2(3\delta \mu + \delta - \mu)]^n[1 + \lambda(2\delta \mu + \delta - \mu)]^2} \quad (4)
\]

Proof. Definition 1 implies that we can write:

\[
\frac{(1 - \lambda) D_{\delta \mu}^n f(z) + \lambda D_{\delta \mu}^{n+1} f(z)}{z} = [s(z)]^\alpha \quad (5)
\]

and

\[
\frac{(1 - \lambda) D_{\delta \mu}^n g(w) + \lambda D_{\delta \mu}^{n+1} g(w)}{w} = [t(w)]^\alpha, \quad (6)
\]
where $s(z), t(w) \in \mathcal{P}$ such that:

$$s(z) = 1 + s_1 z + s_2 z^2 + s_3 z^3 + \cdots, \ (z \in \mathbb{U}) \quad (7)$$

and

$$t(w) = 1 + t_1 w + t_2 w^2 + t_3 w^3 + \cdots, \ (w \in \mathbb{U}). \quad (8)$$

Clearly, we have:

$$[s(z)]^\alpha = 1 + \alpha s_1 z + \left[ \alpha s_2 + \frac{\alpha (\alpha - 1)}{2} s_1^2 \right] z^2 + \cdots$$

and

$$[t(w)]^\alpha = 1 + \alpha t_1 w + \left[ \alpha t_2 + \frac{\alpha (\alpha - 1)}{2} t_1^2 \right] w^2 + \cdots.$$ 

Also, using (1) and (2), we get:

$$\frac{(1 - \lambda)D_{\delta \mu}^n f(z) + \lambda D_{\delta \mu}^{n+1} f(z)}{z} = 1 + [1 + (2\delta \mu + \delta - \mu)]^n [1 + \lambda(2\delta \mu + \delta - \mu)] a_2 z +$$

$$[1 + 2(3\delta \mu + \delta - \mu)]^n [1 + 2\lambda(3\delta \mu + \delta - \mu)] a_3 z^2 + \cdots \quad (9)$$

and

$$\frac{(1 - \lambda)D_{\delta \mu}^n g(w) + \lambda D_{\delta \mu}^{n+1} g(w)}{w} = 1 - [1 + (2\delta \mu + \delta - \mu)]^n [1 + \lambda(2\delta \mu + \delta - \mu)] a_2 w +$$

$$[1 + 2(3\delta \mu + \delta - \mu)]^n [1 + 2\lambda(3\delta \mu + \delta - \mu)] (2a_2^2 - a_3) w^2 + \cdots. \quad (10)$$

Now, equating the coefficients in (5) and (6), we obtain:

$$[1 + (2\delta \mu + \delta - \mu)]^n [1 + \lambda(2\delta \mu + \delta - \mu)] a_2 = \alpha s_1, \quad (11)$$

$$[1 + 2(3\delta \mu + \delta - \mu)]^n [1 + 2\lambda(3\delta \mu + \delta - \mu)] a_3 = \alpha s_2 + \frac{\alpha (\alpha - 1)}{2} s_1^2, \quad (12)$$

$$- [1 + (2\delta \mu + \delta - \mu)]^n [1 + \lambda(2\delta \mu + \delta - \mu)] a_2 = \alpha t_1, \quad (13)$$

$$[1 + 2(3\delta \mu + \delta - \mu)]^n [1 + 2\lambda(3\delta \mu + \delta - \mu)] (2a_2^2 - a_3) = \alpha t_2 + \frac{\alpha (\alpha - 1)}{2} t_1^2. \quad (14)$$

Using (11) and (13), we get:

$$s_1 = -t_1 \quad (15)$$

and

$$2[1 + (2\delta \mu + \delta - \mu)]^{2n} [1 + \lambda(2\delta \mu + \delta - \mu)]^2 a_2^2 = \alpha^2 (s_1^2 + t_1^2). \quad (16)$$
Adding (12) in (14) and then using (16), we obtain:
\[
a_2^2 = \frac{\alpha^2(s_2 + t_2)}{2\alpha[1 + 2(3\delta\mu + \delta - \mu)]^n[1 + 2\lambda(3\delta\mu + \delta - \mu)] - (\alpha - 1)[1 + (2\delta\mu + \delta - \mu)]^{2n}[1 + \lambda(2\delta\mu + \delta - \mu)]^2}
\]

Now, by using Lemma 1, this gives:
\[
|a_2^2| \leq \frac{4\alpha^2}{2\alpha[1 + 2(3\delta\mu + \delta - \mu)]^n[1 + 2\lambda(3\delta\mu + \delta - \mu)] - (\alpha - 1)[1 + (2\delta\mu + \delta - \mu)]^{2n}[1 + \lambda(2\delta\mu + \delta - \mu)]^2},
\]
which proves the result (3). Next, for the estimate on \(|a_3|\), subtracting (14) from (12) in light of (15), we get:
\[
a_3 - a_2^2 = \frac{\alpha(s_2 - t_2)}{2[1 + 2(3\delta\mu + \delta - \mu)]^n[1 + 2\lambda(3\delta\mu + \delta - \mu)]}.
\]
This by using (16), becomes:
\[
a_3 = \frac{\alpha^2(s_t^2 + t_t^2)}{2[1 + (2\delta\mu + \delta - \mu)]^{2n}[1 + \lambda(2\delta\mu + \delta - \mu)]^2 + \alpha(s_2 - t_2)[1 + 2(3\delta\mu + \delta - \mu)]^n[1 + 2\lambda(3\delta\mu + \delta - \mu)].
\]
Finally, by using Lemma 1, we get:
\[
|a_3| \leq \frac{4\alpha^2}{[1 + (2\delta\mu + \delta - \mu)]^{2n}[1 + \lambda(2\delta\mu + \delta - \mu)]^2 + \alpha\frac{[1 + 2(3\delta\mu + \delta - \mu)]^n[1 + 2\lambda(3\delta\mu + \delta - \mu)]}{2\alpha}},
\]
which is the desired result (4). This completes the proof of Theorem 2.

**Definition 2.** A function \(f(z)\) given by (1) is said to be in the class \(\mathcal{H}^{\delta\mu}_{\Sigma}(n, \beta, \lambda)\) if the following conditions are satisfied:
\[
f \in \Sigma, \quad \Re\left\{ \frac{(1 - \lambda)D^n_{\delta\mu}f(z) + \lambda D^{n+1}_{\delta\mu}f(z)}{z} \right\} > \beta
\]
\[ (0 \leq \beta < 1, 0 \leq \mu \leq \delta, \lambda \geq 1, n \in \mathbb{N}_0, z \in \mathbb{U}) \]

and
\[ \Re \left\{ \frac{(1 - \lambda)D_{\delta \mu}^n g(w) + \lambda D_{\delta \mu}^{n+1} g(w)}{w} \right\} > \beta \]
\[ (0 \leq \beta < 1, 0 \leq \mu \leq \delta, \lambda \geq 1, n \in \mathbb{N}_0, w \in \mathbb{U}), \]

where the function \( g \) is given by (2).

Note that in Definition 1 and Definition 2, by putting \( \mu = 0 \) we obtain the classes \( B_{\Sigma}^{\delta, n, \alpha, \lambda} \) and \( H_{\Sigma}^{\delta, n, \beta, \lambda} \) introduced by Patil and Naik [8]; by putting \( \mu = 0, \delta = 1 \) we obtain the classes \( B_{\Sigma}^{\delta, n, \alpha, \lambda} \) and \( H_{\Sigma}^{\delta, n, \beta, \lambda} \) introduced by Porwal and Darus [10]; by putting \( \mu = 0, \delta = 1, n = 0 \) we obtain the classes \( B_{\Sigma}^{\delta, n, \alpha, \lambda} \) and \( H_{\Sigma}^{\delta, n, \beta, \lambda} \) introduced by Frasin and Aouf [5] and by putting \( \mu = 0, \delta = 1, n = 0, \lambda = 1 \) we obtain the classes \( H_{\Sigma}^{\alpha} \) and \( H_{\Sigma}^{\beta} \) introduced by Srivastava et al. [14].

**Theorem 3.** If the function \( f(z) \) given by (1) be in the class \( H_{\Sigma}^{\delta, \beta, \lambda} \), then
\[ |a_2| \leq \sqrt{\frac{2(1 - \beta)}{(1 + 2(3\delta\mu + \delta - \mu))^n[1 + 2\lambda(3\delta\mu + \delta - \mu)]}} \]  
(17)

and
\[ |a_3| \leq \frac{4(1 - \beta)^2}{[1 + (2\delta\mu + \delta - \mu)]^{2n}[1 + \lambda(2\delta\mu + \delta - \mu)][2(1 - \beta)] + 2(1 - \beta)} \]
\[ [1 + 2(3\delta\mu + \delta - \mu)]^{n}[1 + 2\lambda(3\delta\mu + \delta - \mu)]. \]  
(18)

**Proof.** Definition 2 implies that there exists \( s(z), t(w) \in \mathcal{P} \) such that:
\[ \frac{(1 - \lambda)D_{\delta \mu}^n f(z) + \lambda D_{\delta \mu}^{n+1} f(z)}{z} = \beta + (1 - \beta) s(z) \]  
(19)

and
\[ \frac{(1 - \lambda)D_{\delta \mu}^n g(w) + \lambda D_{\delta \mu}^{n+1} g(w)}{w} = \beta + (1 - \beta) t(w), \]  
(20)

where \( s(z) \) and \( t(w) \) are given by (7) and (8) respectively.

See that we have equations (9), (10) and also:
\[ \beta + (1 - \beta) s(z) = 1 + (1 - \beta)s_1z + (1 - \beta)s_2z^2 + \cdots \]

and
\[ \beta + (1 - \beta) t(w) = 1 + (1 - \beta)t_1w + (1 - \beta)t_2w^2 + \cdots. \]
Now, equating the coefficients in (19) and (20), we obtain:
\[
[1 + (2\delta\mu + \delta - \mu)]^n[1 + \lambda(2\delta\mu + \delta - \mu)]a_2 = (1 - \beta)s_1,
\]
\[
[1 + 2(3\delta\mu + \delta - \mu)]^n[1 + 2\lambda(3\delta\mu + \delta - \mu)]a_3 = (1 - \beta)s_2,
\]
\[
- [1 + (2\delta\mu + \delta - \mu)]^n[1 + \lambda(2\delta\mu + \delta - \mu)]a_2 = (1 - \beta)t_1,
\]
\[
[1 + 2(3\delta\mu + \delta - \mu)]^n[1 + 2\lambda(3\delta\mu + \delta - \mu)][2a_2^2 - a_3] = (1 - \beta)t_2.
\]
Using (21) and (23), we obtain:
\[
s_1 = -t_1
\]
and
\[
2[1 + (2\delta\mu + \delta - \mu)]^2n[1 + \lambda(2\delta\mu + \delta - \mu)]^2a_2^2 = (1 - \beta^2)(s_1^2 + t_1^2).
\]
Adding (22) in (24), we obtain:
\[
a_2^2 = \frac{(1 - \beta)(s_2 + t_2)}{2[1 + 2(3\delta\mu + \delta - \mu)]^n[1 + 2\lambda(3\delta\mu + \delta - \mu)]}.
\]
This by using Lemma 1, gives:
\[
|a_2^2| \leq \frac{2(1 - \beta)}{[1 + 2(3\delta\mu + \delta - \mu)]^n[1 + 2\lambda(3\delta\mu + \delta - \mu)]},
\]
which gives the desired result (17). Next, subtracting (24) from (22), we obtain:
\[
2[1 + 2(3\delta\mu + \delta - \mu)]^n[1 + 2\lambda(3\delta\mu + \delta - \mu)](a_3 - a_2^2) = (1 - \beta)(s_2 - t_2)
\]
or
\[
a_3 = a_2^2 + \frac{(1 - \beta)(s_2 - t_2)}{2[1 + 2(3\delta\mu + \delta - \mu)]^n[1 + 2\lambda(3\delta\mu + \delta - \mu)]}.
\]
Using (25), this becomes:
\[
a_3 = \frac{(1 - \beta)^2(s_2^2 + t_2^2)}{2[1 + (2\delta\mu + \delta - \mu)]^2n[1 + \lambda(2\delta\mu + \delta - \mu)]^2} + \frac{(1 - \beta)(s_2 - t_2)}{2[1 + 2(3\delta\mu + \delta - \mu)]^n[1 + 2\lambda(3\delta\mu + \delta - \mu)]}.
\]
This by using Lemma 1, yields:
\[
|a_3| \leq \frac{4(1 - \beta^2)}{[1 + (2\delta\mu + \delta - \mu)]^2n[1 + \lambda(2\delta\mu + \delta - \mu)]^2} + \frac{2(1 - \beta)}{2[1 + 2(3\delta\mu + \delta - \mu)]^n[1 + 2\lambda(3\delta\mu + \delta - \mu)]},
\]
which is the desired result (18). This completes the proof of Theorem 3.
3. Conclusions

- If we put $\mu = 0$ in Theorem 2 and Theorem 3; we obtain Theorem 5 and Theorem 7 given by Patil and Naik [8].

- If we put $\mu = 0$ and $\delta = 1$ in Theorem 2 and Theorem 3; we obtain Theorem 2.1 and Theorem 3.1 given by Porwal and Darus [10].

- If we put $\mu = 0$, $\delta = 1$ and $n = 0$ in Theorem 2 and Theorem 3; we obtain Theorem 2.2 and Theorem 3.2 given by Frasin and Aouf [5].

- If we put $\mu = 0$, $\delta = 1$, $n = 0$ and $\lambda = 1$ in Theorem 2 and Theorem 3; we obtain Theorem 1 and Theorem 2 given by Srivastava et al.[14].

References


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