UNIVALENCY OF SOME OPERATORS FOR ANALYTIC FUNCTIONS

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ABSTRACT. For analytic functions $f(z)$ in the open unit disk $U$, univalency of some integral operators concerning with Alexander type integrals is considered. Also some subordinations for analytic functions $f(z)$ in $U$ are discussed with the Schwarzian derivative of $f(z)$.

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1. Introduction

Let $\mathcal{H}$ denote the class of functions $f(z)$ which are analytic in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$. Also let $\mathcal{A}$ be the subclass of $\mathcal{H}$ consisting of functions $f(z)$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (z \in U).$$

Let $\mathcal{S}$ be the subclass of $\mathcal{A}$ consisting of $f(z)$ which are univalent in $U$. If $f(z) \in \mathcal{A}$ satisfies

$$\text{Re} \left( \frac{zf''(z)}{f'(z)} \right) > \alpha \quad (z \in U)$$

for some real $\alpha (0 \leq \alpha < 1)$, then $f(z)$ is said to be starlike of order $\alpha$ in $U$ and denoted by $f(z) \in \mathcal{S}^*(\alpha)$. For $\alpha = 0$, we say that $f(z) \in \mathcal{S}^*$ is starlike with respect to the origin. Further, if a function $f(z) \in \mathcal{A}$ satisfies $zf''(z) \in \mathcal{S}^*(\alpha) (0 \leq \alpha < 1)$, then $f(z)$ is said to be convex of order $\alpha$ in $U$ and denoted by $f(z) \in \mathcal{K}(\alpha)$. A function $f(z) \in \mathcal{K}(\alpha)$ satisfies

$$\text{Re} \left( 1 + \frac{zf''(z)}{f'(z)} \right) > \alpha \quad (z \in U).$$
For $\alpha = 0$, we write that $K(0) = K$. We note that

$$K(\alpha) \subset S^*(\alpha) \subset S \subset A \subset H.$$ 

If there exists a function $g(z) \in K$ such that

$$\text{Re} \left( e^{-i\beta} \frac{f'(z)}{g(z)} \right) > 0 \quad (z \in \mathbb{U})$$

for $\beta \in (-\pi/2, \pi/2)$ and $f(z) \in A$, then $f(z)$ is said to be close-to-convex in $\mathbb{U}$ and denoted by $f(z) \in C$. It is known that $C \subset S$.

For $f(z) \in H$, the Schwarzian derivative of $f(z)$ is given by

$$\{f; z\} = 6 \left( \frac{\partial^2}{\partial z^2} \log \left( \frac{f(z) - f(\zeta)}{z - \zeta} \right) \right)_{z=\zeta} = \left( \frac{f''(z)}{f'(z)} \right)' - \frac{1}{2} \left( \frac{f''(z)}{f'(z)} \right)^2.$$ 

For the Schwarzian derivative $\{f; z\}$ for $f(z) \in H$, it is well-known that if $f(z) \in H$ is univalent in $\mathbb{U}$, then

$$|\{f; z\}| \leq \frac{6}{(1 - |z|^2)^2} \quad (z \in \mathbb{U})$$

and the equality holds true for the Koebe function $f(z) = z/(1 - z)^2$. Further, we know that the Nehari’s condition (see Nehari [10])

$$|\{f; z\}| \leq \frac{2}{(1 - |z|^2)^2} \quad (z \in \mathbb{U})$$

implies that $f(z) \in H$ is univalent in $\mathbb{U}$.

Note that $f(z) \in A$ is uniformly locally univalent if and only if the pre-Schwarzian derivative

$$T_f(z) = \frac{f''(z)}{f'(z)}$$

is hyperbolically bounded, that is, that the norm

$$\| f \| = \sup_{|z|<1} (1 - |z|^2)|T_f(z)|$$

is finite. This quantity can be regarded as the Bloch norm of function $(\log f(z))'$. Both of the pre-Schwarzian derivative and the norm $\| f \|$ play a central role in the theory of Teichmüller spaces, inner radius of univalence, quasiconformal extension, etc. If $f(z) \in A$ is univalent in $\mathbb{U}$, then $\| f \| < 6$ and the bound 6 is sharp for the
Koebe function \( k(z) = z/(1-z)^2 \).
Conversely, if \( f(z) \in \mathcal{A} \) satisfies \( \| f \| < 1 \), then \( f(z) \) is univalent in \( U \) by Becker [1]. Also, it is known that \( \| f \| < 4 \) for \( f(z) \in \mathcal{K} \). For \( f(z) \in \mathcal{A} \), the Alexander transformation \( J[f](z) \) is defined by
\[
J[f](z) = \int_0^z \frac{f(t)}{t} dt.
\]
If \( f(z) \in \mathcal{S} \), then \( f(z) \) is locally univalent and \( \| J[f] \| < 6 \) by Kim, Choi and Sugawa [6]. Also, Yamashita [12] proved that if \( f(z) \in \mathcal{S}^* \), then \( \| J[f] \| < 6 - 4\alpha \) and \( \| J[f] \| < 4(1 - \alpha) \). By means of (1.5) and (1.8), we see that
\[
\{ f; z \} = (T_f(z))' - \frac{1}{2}(T_f(z))^2.
\]
The Alexander transformation \( J[f](z) \) of \( f(z) \in \mathcal{A} \) is also called as Biernacki’s integral. It is known that \( J[f](\mathcal{S}^*) = \mathcal{K} \) while \( J[f](\mathcal{S}) \) is not in \( \mathcal{S} \). In this paper, we would like to extend the type of functions \( f(z) \) to be considered by introducing a parameter \( \alpha \) and setting an integral of the form
\[
F_\alpha(z) = \int_0^z \left( \frac{tf'(t)}{f(t)} \right)^\alpha dt.
\]
For more details on this integral, we refer to Goodman [4]. The following lemma due to Fukui and Sakaguchi [3] is a generalization of Jack’s lemma by Jack [5] (also by Miller and Mocanu [9]).

**Lemma 1.1** Let \( w(z) = a_p z^p + a_{p+1} z^{p+1} + \cdots \) be analytic in \( U \) with \( a_p \neq 0 \) and \( p \geq 1 \). If the maximum value of \( |w(z)| \) on the circle \( |z| = r < 1 \) is attained at \( z = z_0 \), then \( z_0 w'(z_0)/w(z_0) \) is real and
\[
\frac{z_0 w'(z_0)}{w(z_0)} \geq p.
\]

2. Univalency of some operators

We first derive

**Theorem 2.1** Let \( f(z) \) be analytic in \( U \) with \( f(0) = 0 \). If \( f(z) \) satisfies
\[
|f(z)| \leq \frac{M}{1-|z|^2} \quad (z \in U)
\]
for a bounded positive constant $M$, then

\begin{equation}
|f(z)| \leq \frac{3\sqrt{3}M|z|}{2} \leq \frac{3\sqrt{3}|z|}{2(1-|z|^2)} \quad (|z| \leq \frac{\sqrt{3}}{3})
\end{equation}

and

\begin{equation}
|f(z)| \leq \frac{\sqrt{3}M|z|}{1-|z|^2} \leq \frac{3\sqrt{3}M|z|}{2(1-|z|^2)} \quad \left(\frac{\sqrt{3}}{3} \leq |z| < 1\right).
\end{equation}

**Proof** For the case of $|z| \leq \sqrt{3}/3$, we have

\begin{equation}
\frac{1}{1-|z|^2} \leq \frac{3}{2}.
\end{equation}

Thus, the inequality (2.1) gives

\begin{equation}
|f(z)| \leq \frac{3M}{2} \quad (|z| \leq \frac{\sqrt{3}}{3}).
\end{equation}

Therefore, applying the Schwarz lemma for $f(z)$ with $|z| \leq \sqrt{3}/3$, we obtain that

\begin{equation}
|f(z)| \leq \sqrt{3}|z| \frac{3M}{2} \quad (|z| \leq \frac{\sqrt{3}}{3})
\end{equation}

which shows (2.2). If $\sqrt{3}/3 \leq |z| < 1$, we know that $\sqrt{3}|z| \geq 1$. This gives us that

\begin{equation}
|f(z)| \leq \frac{\sqrt{3}M|z|}{1-|z|^2} \quad \left(\frac{\sqrt{3}}{3} \leq |z| < 1\right)
\end{equation}

which implies the inequality (2.3).

**Corollary 2.1** If $f(z)$ is analytic in $U$ with $f(0) = 0$, then there exists some $z \in U$ such that

\begin{equation}
|f(z)| \leq \frac{M}{1-|z|^2}
\end{equation}

satisfies

\begin{equation}
|f(z)| \leq \frac{3\sqrt{3}M|z|}{2(1-|z|^2)}
\end{equation}

for a positive constant $M$. 

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Remark 2.1 Noting that \(3\sqrt{3}/2 = 2.598\ldots\), we conjecture that \(3\sqrt{3}/2\) in Corollary 2.1 can be replaced by 1.

Next, we derive

**Theorem 2.2** For a function \(f(z) \in S\), we assume that the function \((zf'(z)/f(z))^\alpha\) is analytic in \(U\) for \(\alpha > 0\) with

\[
(2.10) \quad \left. \left( \frac{zf'(z)}{f(z)} \right)^\alpha \right|_{z=0} = 1.
\]

Then, the integral transformation \(F_\alpha(z)\) defined by (1.12) is univalent in \(U\) for

\[
(2.11) \quad 0 < \alpha \leq \alpha_0 = \frac{2\sqrt{5} - 4}{15\sqrt{3}} = 0.0181725\ldots.
\]

**Proof** Note that

\[
(2.12) \quad F_\alpha'(z) = \left( \frac{zf'(z)}{f(z)} \right)^\alpha \quad (z \in U)
\]

by \(F_\alpha(z)\) in (1.12). This gives us that

\[
(2.13) \quad \frac{F_\alpha''(z)}{F_\alpha'(z)} = \frac{\alpha}{z} \left( 1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} \right).
\]

If we put

\[
(2.14) \quad h(z) = 1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} \quad (z \in U),
\]

we have that \(h(0) = 0\) and

\[
(2.15) \quad |h(z)| \leq \left| 1 + \frac{zf''(z)}{f'(z)} \right| + \left| \frac{zf'(z)}{f(z)} \right|.
\]

On the other hand, it is well-known that if \(f(z) \in S\), then

\[
(2.16) \quad \left| \frac{zf''(z)}{f'(z)} - \frac{2|z|^2}{1 - |z|^2} \right| \leq \frac{4|z|}{1 - |z|^2} \quad (z \in U)
\]

that is,

\[
(2.17) \quad \left| 1 + \frac{zf''(z)}{f'(z)} - \frac{1 + |z|^2}{1 - |z|^2} \right| \leq \frac{4|z|}{1 - |z|^2} \quad (z \in U).
\]
This gives that
\[(2.18) \quad \left| 1 + \frac{zf''(z)}{f'(z)} \right| \leq \frac{4|z|}{1 - |z|^2} + \frac{1 + |z|^2}{1 - |z|^2} < \frac{6}{1 - |z|^2} \quad (z \in \mathbb{U}).\]

Further, we know that
\[(2.19) \quad \left| \frac{zf'(z)}{f(z)} \right| \leq \frac{1 + |z|}{1 - |z|} = \frac{(1 + |z|)^2}{1 - |z|^2} < \frac{4}{1 - |z|^2} \quad (z \in \mathbb{U}).\]

Therefore, the inequality (2.15) implies that
\[(2.20) \quad |h(z)| < \frac{10}{1 - |z|^2} \quad (z \in \mathbb{U}).\]

Considering $M = 10$ in (2.1) of Theorem 2.1, we say that
\[(2.21) \quad |h(z)| < \frac{15\sqrt{3}|z|}{1 - |z|^2} \quad (z \in \mathbb{U}).\]

Therefore, we have that
\[(2.22) \quad \left| \frac{F''_{\alpha}(z)}{F'_{\alpha}(z)} \right| \leq \frac{\alpha}{|z|} |h(z)| < \frac{15\sqrt{3}\alpha}{1 - |z|^2} \quad (z \in \mathbb{U}).\]

By using of the result in [11], we know that there exists a point $z \in \mathbb{U}$ that if
\[(2.23) \quad |h(z)| < \frac{1}{1 - |z|^2} \quad (z \in \mathbb{U}),\]

then
\[(2.24) \quad |h'(z)| < \frac{4}{(1 - |z|^2)^2} \quad (z \in \mathbb{U}).\]

It follows from the above that
\[(2.25) \quad \left| \left( \frac{F''_{\alpha}(z)}{F'_{\alpha}(z)} \right)' \right| < \frac{60\sqrt{3}\alpha}{(1 - |z|^2)^2} \quad (z \in \mathbb{U}).\]

Therefore, we have that
\[(2.26) \quad |\{F_{\alpha}(z); z\}| \leq \left| \left( \frac{F''_{\alpha}(z)}{F'_{\alpha}(z)} \right)' \right| + \frac{1}{2} \left( \frac{F''_{\alpha}(z)}{F'_{\alpha}(z)} \right)^2 \]

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\[ \leq \frac{60\sqrt{3}\alpha}{(1-|z|^2)^2} + \frac{1}{2} \left( \frac{15\sqrt{3}\alpha}{1-|z|^2} \right)^2 = \frac{15(45\alpha + 8\sqrt{3})\alpha}{2(1-|z|^2)^2} \quad (z \in \mathbb{U}). \]

Applying the Nehari’s condition (1.7) for \( F_{\alpha}(z) \), we need that
\[ (2.27) \quad \frac{15(45\alpha + 8\sqrt{3})\alpha}{2} \leq 2, \]
that is, that
\[ (2.28) \quad 0 < \alpha \leq \alpha_0 = \frac{2\sqrt{5} - 4}{15\sqrt{3}} = 0.0181725 \ldots. \]
This completes the proof of the theorem.

Next, we recall here a result by Chichra and Singh [2] that if
\[ (2.29) \quad z + z^2\log \frac{g(z)}{z} \in S^*, \]
then there exist some \( t \) (\( 0 \leq t \leq 1 \)) and \( \alpha \) (\( 0 \leq \alpha \leq 1/2 \)) such that
\[ (2.30) \quad tz + (1-t) \int_0^z \left( \frac{tg'(t)}{g(t)} \right)^\alpha dt \in S^*. \]
Letting
\[ (2.31) \quad \frac{g(z)}{z} = \frac{zf'(z)}{f(z)} \]
for \( f(z) \in A \), Theorem 2.2 becomes

**Theorem 2.3** Assume that \( g(z) \in A \) satisfies
\[ (2.32) \quad z\exp \left( \int_0^z \frac{g(t)}{t} - \frac{1}{t} dt \right) \in S, \]
the function \( (g(z)/z)^\alpha \) is analytic in \( \mathbb{U} \) with \( 0 < \alpha < 1 \) and
\[ (2.33) \quad \left. \left( \frac{g(z)}{z} \right)^\alpha \right|_{z=0} = 1. \]
If \( 0 < \alpha \leq \alpha_0 = (2\sqrt{5} - 4)/15\sqrt{3} = 0.0181725 \ldots \), then the integration \( \int_0^z (g(t)/t)^\alpha dt \) is univalent in \( \mathbb{U} \).
By means of the result due to Krzyż [7], we know that \( g(z) \in S \) is not implies that \( \int_0^z (g(t)/t) \, dt \in S \). The counterexample for the above is given by

\[
(2.34) \quad g(z) = \frac{z}{(1 - iz)^{1-i}}.
\]

On the other hand, Merkes and Wright [8] showed that if \( g(z) \in S^* \), then

\[
(2.35) \quad \int_0^z \left( \frac{g(t)}{t} \right) ^\alpha \, dt \in C
\]

for \(-1/2 \leq \alpha \leq 3/2\). Theorem 2.3 says that if

\[
(2.36) \quad z \exp \left( \int_0^z \frac{g(t)}{t} - \frac{1}{t} \, dt \right) \in S,
\]

then

\[
(2.37) \quad \int_0^z \left( \frac{g(t)}{t} \right) ^\alpha \, dt \in S
\]

for \( 0 < \alpha \leq \alpha_0 = (2\sqrt{5} - 4)/15\sqrt{3} \).

**Corollary 2.2** If \( g(z) \in A \) satisfies

\[
(2.38) \quad \text{Re} \left( \frac{g(z)}{z} \right) > 0 \quad (z \in \mathbb{U}),
\]

then

\[
(2.39) \quad \int_0^z \left( \frac{g(t)}{t} \right) ^\alpha \, dt
\]

is univalent in \( \mathbb{U} \), where \( 0 < \alpha \leq \alpha_0 = (2\sqrt{5} - 4)/15\sqrt{3} \).

3. An application of Schwarzian derivative

Next, we would like to consider an application of Schwarzian derivative concerning with the subordinations. Let \( f(z) \in A \) and \( g(z) \in A \). Then the function \( f(z) \) is said to subordinate to \( g(z) \) if there exists a function \( w(z) \) analytic in \( \mathbb{U} \) with \( w(0) = 0 \) and \( |w(z)| < 1 \) such that \( f(z) = g(w(z)) \) for \( z \in \mathbb{U} \). We write that

\[
(3.1) \quad f(z) \prec g(z) \quad (z \in \mathbb{U})
\]
if \( f(z) \) subordinates to \( g(z) \) in \( U \). Also, if \( g(z) \) is univalent in \( U \), then \( f(z) \prec g(z) \) is equivalent to \( f(0) = g(0) \) and \( f(U) \subset g(U) \) (see Miller and Mocanu [9]).

Now, we derive

**Theorem 3.1** Let \( f(z) \in \mathcal{A} \) satisfy

\[
|z^2 \{ f ; z \} | < \alpha (1 - \beta) \quad (z \in \mathbb{U}),
\]

where \( 0 < \alpha < 1 \) and

\[
\left| \frac{zh''(z)}{h'(z)} - \frac{2zh'(z)}{h(z) + 1} \right| \leq \beta \quad (z \in \mathbb{U})
\]

with

\[
h(z) = (f'(z))^{1/\alpha} \neq \pm 1.
\]

Then we have that

\[
f'(z) \prec \left( \frac{1 + z}{1 - z} \right)^\alpha \quad (z \in \mathbb{U})
\]

or

\[
| \arg f'(z) | < \frac{\pi}{2\alpha} \quad (z \in \mathbb{U}).
\]

Therefore, \( f(z) \) is univalent in \( \mathbb{U} \).

**Proof** For \( h(z) = (f'(z))^{1/\alpha} \) \( (0 < \alpha < 1) \), we define the function \( w(z) \) by

\[
w(z) = \frac{h(z) - 1}{h(z) + 1} = \frac{c_n}{2} z + \cdots
\]

with \( w(0) = 0 \). This implies that

\[
f'(z) = \left( \frac{1 + w(z)}{1 - w(z)} \right)^\alpha.
\]

It follows from (3.8) that

\[
f''(z) = \frac{2\alpha w'(z)}{1 - w(z)^2} \left( \frac{1 + w(z)}{1 - w(z)} \right)^\alpha = \frac{2\alpha w'(z)}{1 - w(z)^2} f'(z),
\]
that is, that

\begin{equation}
\frac{f''(z)}{f'(z)} = \frac{2\alpha w'(z)}{1 - w(z)^2}.
\end{equation}

Thus, we obtain that

\begin{equation}
\left( \frac{f''(z)}{f'(z)} \right)^2 = \left( \frac{zf''(z)}{f'(z)} \right)^2 \frac{1}{z^2} = \left( \frac{2\alpha z w'(z)}{1 - w(z)^2} \right)^2 \frac{1}{z^2}.
\end{equation}

We suppose that there exists a point \( z_0 \in \mathbb{U} \) such that \(|w(z)| < 1 (|z| < |z_0| < 1)\) and \(|w(z_0)| = 1\). Then Lemma 1.1 gives us that

\begin{equation}
\frac{z_0 w'(z_0)}{w(z_0)} = k \geq 1.
\end{equation}

Further, by the result due to Miller and Mocanu [9], we have that

\begin{equation}
\Re \left( \frac{z_0 w''(z_0)}{w'(z_0)} \right) \geq 0.
\end{equation}

Therefore, we have that

\begin{equation}
\left( \frac{f''(z)}{f'(z)} \right)^2 = \left( \frac{2\alpha k w(z_0)}{1 - w(z_0)^2} \right)^2 \frac{1}{z_0^2} = \left( \frac{i\alpha k}{\sin \theta} \right)^2 \frac{1}{z_0^2} = - \left( \frac{\alpha k}{\sin \theta} \right)^2 \frac{1}{z_0^2},
\end{equation}

where \( w(z_0) = e^{i\theta} \) (\( 0 \leq \theta < 2\pi \)).

Also, we see that

\begin{equation}
\left. \left( \frac{f''(z)}{f'(z)} \right)' \right|_{z=z_0} = \left. \left( \frac{2\alpha w'(z)}{1 - w(z)^2} \right)' \right|_{z=z_0}
\end{equation}

\begin{align*}
&= 2\alpha \left( \frac{w''(z_0)}{1 - w(z_0)^2} \right) + \left. \frac{4\alpha w(z)(w'(z))^2}{(1 - w(z)^2)^2} \right|_{z=z_0} \\
&= ik\alpha \left( \frac{z_0 w''(z_0)}{w'(z_0)} \right) \frac{1}{z_0} + \left( \frac{ik}{\sin \theta} \right)^2 \frac{\alpha w(z_0)}{z_0^2} \\
&= \frac{k\alpha}{\sin \theta} \left\{ i \left( \frac{z_0 w''(z_0)}{w'(z_0)} \right) - \frac{kw(z_0)}{\sin \theta} \right\} \frac{1}{z_0^2}.
\end{align*}
Consequently, we obtain that

$$z_0^2 \{f; z\} = \frac{k\alpha}{\sin \theta} \left\{ \frac{1}{2i} \left( \frac{z_0 w''(z_0)}{w'(z_0)} \right) - \frac{kw(z_0)}{\sin \theta} + \frac{\alpha k}{2\sin \theta} \right\}$$

and so

$$|z_0^2 \{f; z_0\}| \geq \frac{\alpha}{2} \left| \frac{k}{\sin \theta} \frac{\alpha - 2\cos \theta - 2i\sin \theta}{w'(z_0)} \right| + 2 \left| \frac{z_0 w''(z_0)}{w'(z_0)} \right|.$$

If we define a function $p(x)$ by

$$p(x) = \frac{x^2 - 4\alpha x + 4}{1 - x^2} \quad (x = \cos \theta),$$

then

$$p'(x) = \frac{-2(2x - \alpha)(\alpha x - 2)}{(1 - x^2)^2}$$

gives that $p(x)$ takes its minimum value at $x = \alpha/2 < 1/2$, because $0 < \alpha < 1$ and $-1 \leq x \leq 1$. This shows us that $p(x) \geq 4$ and so

$$|z_0^2 \{f; z_0\}| \geq \frac{\alpha}{2} \frac{\alpha - 2\cos \theta + 4}{1 - \cos^2 \theta} - 2 \left| \frac{z_0 w''(z_0)}{w'(z_0)} \right| \geq \alpha (1 - \beta).$$

This contradicts the condition (3.2) of the theorem. Therefore, there is no $z_0 \in U$ such that $|w(z_0)| = 1$. This implies that there exists $w(z)$ such that

$$f'(z) = \left( \frac{1 + w(z)}{1 - w(z)} \right)^\alpha \quad (z \in U)$$

with $w(0) = 0$ and $|w(z)| < 1 (z \in U)$. Consequently, we prove the subordination (3.5).

Further, since

$$|\arg \left( \frac{1 + z}{1 - z} \right) | < \frac{\pi}{2} \quad (z \in U),$$
we obtain (3.6) for $\arg f'(z)$.

Making $\alpha = 1/2$ in Theorem 3.1, we derive

**Corollary 3.1** Let $f(z) \in A$ satisfy

$$|z^2\{f; z\}| < \frac{1-\beta}{2} \quad (z \in \mathbb{U})$$

with

$$\frac{zh''(z)}{h'(z)} - \frac{2zh'(z)}{h(z) + 1} \leq \beta \quad (z \in \mathbb{U})$$

and $h(z) = \sqrt{f'(z)} \neq \pm 1$. Then we have

$$f'(z) < \sqrt{\frac{1+z}{1-z}} \quad (z \in \mathbb{U})$$

or

$$|\arg f'(z)| < \frac{\pi}{4} \quad (z \in \mathbb{U}).$$

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