SOME RESULTS ON AN EQUIVALENCE RELATION ON THE SET OF CLOSED AND BOUNDED VALUED MULTIFUNCTIONS

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Abstract. By using the notion of the fixed point set of multi-valued mappings, we introduce an equivalence relation on the set of all closed and bounded valued multifunction on a metric space. By using the notion we provide some related results.

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1. Introduction

In 1966, Sam Bernard Jr. Nadler finished his Ph.D. thesis on differential analysis in university of Georgia ([2]). Later, he published some works about results of his thesis ([3], [4] and [6]). He interested fixed point theory by starting basic notions of fixed points and contractive mappings ([5], [7] and [8]). In 1969, he started study of fixed points of multivalued contractive mappings ([9]). In 1970, he published his most famous work in this area ([10]). Hereafter, many researchers reviewed common fixed points of different types of multivalued contractions (see for example, [11], [12] and [13]). In this paper, we introduce an equivalence relation on the set of all closed and bounded valued multifunction on a metric space. Also by using the notion, we provide some related results.

Let \( X \) be a nonempty set, \( \mathcal{P}(X) \) the set of all nonempty subsets of \( X \), \( T \) a multi-valued mapping on \( X \) into \( \mathcal{P}(X) \) and \( \mathcal{F}_T \) the fixed point set of \( T \), that is, \( \mathcal{F}_T = \{ x \in X : x \in Tx \} \). For a topological space \( (Y, \tau) \), we denote the set of all nonempty closed subsets of \( Y \) by \( \mathcal{P}_{cl}(Y) \) and the set of all nonempty closed and bounded subsets of \( Y \) by \( \mathcal{P}_{b,cl}(Y) \) whenever \( Y \) is a metric space.

Let \( (X, d) \) be a metric space, \( x \in X \) and \( A, B \subseteq X \). It is well-known that \( D(x, A) = \inf_{y \in A} d(x, y) \), \( H(A, B) = \max\{\sup_{x \in A} D(x, B), \sup_{y \in B} D(y, A)\} \) and \( \delta(A, B) = \sup\{d(a, b) : a \in A, b \in B\} \). Then, \( H \) is a metric on closed bounded subsets of \( X \) which is called the Hausdorff metric.
2. Main results

Let \((X,d)\) be a metric space. Denote by \(F\) the set of all multi-valued mappings on \(X\) into \(P_{b,cl}(X)\). Define the relation \(\sim\) on \(F\) by \(F \sim G\) whenever \(\tilde{\mathcal{F}}_F = \tilde{\mathcal{F}}_G\) for all \(F,G \in F\). One can check that \(\sim\) is an equivalence relation on \(F\). Denote by \(\tilde{F}\) the equivalence classes of \(F\), that is, \(\tilde{F} = \frac{F}{\sim} = \{\tilde{F} : F \in F\}\). Also, define \(\tilde{d} : \tilde{F} \times \tilde{F} \to [0,\infty)\) by \(\tilde{d}(\tilde{F},\tilde{G}) = H(\tilde{\mathcal{F}}_F, \tilde{\mathcal{F}}_G)\). It is easy to see that \((\tilde{F},\tilde{d})\) is a metric space.

Note that, there is a connection between common fixed points of two multivalued mappings and \(F\) whenever \(S \in \tilde{T}\).

**Lemma 2.1.** Let \((X,d)\) be a metric space, \(m \geq 1\), \(c > 1\) and \(S,T : X \to P_{b,cl}(X)\) two multi-valued mappings such that \(\tilde{\mathcal{F}}_S \neq \emptyset\). Suppose that for each \(x \in X\) and \(y \in Sx\) (or \(y \in Tx\)) there exists \(z \in Ty\) (respectively \(z \in Sy\)) such that

\[
d^{3m}(x, y) - \frac{3}{4\sqrt{4}} c^2 d^{2m}(y, z) d(x, y) - \frac{c^3}{8} d^{3m}(y, z) \geq 0. \tag{1}\]

Then \(\tilde{\mathcal{F}}_T \neq \emptyset\) and \(\tilde{S} = \tilde{T}\).

**Proof.** Let \(u \in \tilde{\mathcal{F}}_S\) and \(z \in Tu\). By using the relation (1), we get

\[
d^3(u, u) - \frac{3}{4\sqrt{4}} c^2 d^2(u, z) d(u, u) - \frac{c^3}{8} d^3(u, z) \geq 0.
\]

Hence, \(-\frac{c^3}{8} d^3(u, z) \geq 0\) and so \(d(u, z) = 0\). This implies that \(z = u\) and so \(u \in Tu\).

Thus, \(\tilde{\mathcal{F}}_T \neq \emptyset\) and \(\tilde{\mathcal{F}}_S \subseteq \tilde{\mathcal{F}}_T\). A similar proof shows that \(\tilde{\mathcal{F}}_T \subset \tilde{\mathcal{F}}_S\). Therefore, \(\tilde{S} = \tilde{T}\).

Let \((X,d)\) be a metric space and \(V : X \to P_{b,cl}(X)\) a multi-valued map. We say that \(T\) has the property \((M)\) whenever for each convergent sequence \(\{x_n\}_{n \geq 0}\) with \(x_n \to x\) and \(x_{2n-1} \in Tx_{2n-2}\) for all \(n\) (or \(x_{2n} \in TV x_{2n-1}\) for all \(n\)) we have \(x \in Tx\).

**Theorem 2.2.** Let \((X,d)\) be a complete metric space, \(S,T : X \to P_{b,cl}(X)\) two multi-valued mappings, \(m \geq 1\) and \(c > 1\). Suppose that for each \(x \in X\) and \(y \in Sx\) (or \(y \in Tx\)) there exists \(z \in Ty\) (respectively \(z \in Sy\)) such that

\[
d^{3m}(x, y) - \frac{3}{4\sqrt{4}} c^2 d^{2m}(y, z) d(x, y) - \frac{c^3}{8} d^{3m}(y, z) \geq 0.
\]

If one of the multi-valued mappings \(S\) and \(T\) have the property \((M)\), then \(\tilde{S} = \tilde{T}\).

**Proof.** Let \(x_0 \in X\) be an arbitrary element and \(x_1 \in Sx_0\). Choose \(x_2 \in Tx_1\) such that \(d^{3m}(x_0, x_1) - \frac{3}{4\sqrt{4}} c^2 d^{2m}(x_1, x_2) d(x_0, x_1) - \frac{c^3}{8} d^{3m}(x_1, x_2) \geq 0\). There exists
$x_3 \in Sx_2$ such that $d^{3m}d(x_1, x_2) - \frac{3}{4}c^2d^{2m}(x_2, x_3)d(x_1, x_2) - \frac{c^3}{8}d^{3m}(x_2, x_3) \geq 0$. By continuing this process we obtain a sequence $\{x_n\}_{n \geq 0}$ in $X$ such that $x_{2n-1} \in Sx_{2n-1}$ and $x_{2n} \in Tx_{2n-1}$ for all $n$ and

$$d^{3m}(x_n, x_{n-1}) - \frac{3}{4}c^2d^{2m}(x_n, x_{n+1})d(x_n, x_{n-1}) - \frac{c^3}{8}d^{3m}(x_n, x_{n+1}) \geq 0$$

for all $n$. Note that, the inequality (2) is a third degree polynomial in the variable $d^{m}(x_n, x_{n-1})$ with the discriminant

$$\Delta = 4\left(\frac{-3}{4}c^2d^{2m}(x_n, x_{n+1})\right)^3 + 27\left(\frac{-c^3}{8}d^{3m}(x_n, x_{n+1})\right)^2.$$ 

Thus, $d^{m}(x_n, x_{n-1}) \geq -2\sqrt[3]{\frac{c^3}{8}d^{3m}(x_n, x_{n+1})} = cd^{m}(x_n, x_{n+1})$. If $k^m = \frac{1}{c}$, then we obtain $k < 1$ and $0 \leq d^{m}(x_n, x_{n+1}) < k^md^{m}(x_n, x_{n-1})$. This implies that $d(x_n, x_{n-1}) \leq kd(x_{n-1}, x_n)$ for all $n$. Hence, $d(x_n, x_{n+1}) \leq k^nd(x_0, x_1)$ for all $n$. It is easy to see that $d(x_n, x_{n+p}) \leq \frac{k^p}{1-k}d(x_0, x_1)$ for all $n$ and $p$. Thus, $\{x_n\}_{n \geq 0}$ is a Cauchy sequence in $X$. Choose $u \in X$ such that $x_n \to u$. Since $x_{2n-1} \in Sx_{2n-1}$ and $x_{2n} \in Tx_{2n-1}$ for all $n$ and one of the multi-valued mappings $S$ and $T$ have the property $(M)$, we conclude that $u \in Su$ or $u \in Tu$. By using Lemma 2.1, we get $S = T$.

We need the followings for our last result.

**Lemma 2.3.** [13] Let $(X, d)$ be a metric space, $A$ and $B$ two bounded subsets of $X$ and $k > 1$. Then for each $a \in A$ there exists $b \in B$ such that $d(a, b) \leq kH(A, B)$.

This implies easily next Lemma.

**Lemma 2.4.** [13] Let $(X, d)$ be a metric space, $k > 1$ and $S, T : X \to P_{d,b}(X)$ two multi-valued mappings. Then for each $y \in Sx$ (or $y \in Tx$) there exists $z \in Ty$ (respectively $z \in Sx$) such that $d(y, z) \leq kH(Sx, Ty)$.

**Theorem 2.5.** Let $(X, d)$ be a complete metric space, $T_1, T_2 : X \to P_{b,cl}(X)$ two multi-valued mappings, $m \geq 1$ and $c > 1$. Suppose that for each $x, y \in X$ with $c^2\delta^{2m}(y, T_2y) + 6c\delta^{m}(y, T_2y)\delta^{m}(x, T_1x) + 8\delta^{2m}(x, T_1x) \neq 0$ we have

$$H^m(T_1x, T_2y) \leq \frac{8d^{3m}(x, T_1x)}{c^2\delta^{2m}(y, T_2y) + 6c\delta^{m}(y, T_2y)\delta^{m}(x, T_1x) + 8\delta^{2m}(x, T_1x)}.$$ 

(3)

Then $\tilde{T}_1 = \tilde{T}_2$. 

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Proof. By using the inequality (3), we obtain
\[ H_m(T_1x, T_2y)(c^2 \delta^m(y, T_2y) + 6c^m(y, T_2y)\delta^m(x, T_1x) + 8\delta^m(x, T_1x)) \leq 8\delta^m(x, T_1x) \]
for all \( x \in X \) and \( y \in T_1x \). Let \( 1 < c < k^m, x \in X \) and \( y \in T_1x \). By using Lemma 2.4, there exists \( z \in T_2y \) such that \( d(y, z) \leq kH(T_1x, T_2y) \). Hence,
\[ cd^m(y, z)(c^2d^m(y, z) + \frac{6cd^m(y, z)d^m(x, y)}{\sqrt{4}})d^m(x, y) \leq 8d^m(x, y). \]
Thus for each \( x \in X \) and \( y \in T_1x \) there exists \( z \in T_2y \) such that
\[ d^m(x, y) - \frac{3}{4\sqrt{4}} cd^m(y, z)d^m(x, y) - \frac{c^3}{8}d^3(y, z) \geq 0. \]
Now, we show that \( T_1 \) has the property (M). Let \( (x_n)_{n \geq 0} \) be a convergent sequence in \( X \) with \( x_n \to x, x_{2n-1} \in T_1x_{2n-2} \) and \( x_{2n} \in T_2x_{2n-1} \) for all \( n \). Then, we have
\[ d(T_1x, x_{2n}) \leq H(T_1x, T_2x_{2n-1}) \]
for all \( n \). Hence,
\[ cd^m(T_1x, x_{2n})(c^2d^m(x_{2n-1}, x_{2n}) + 6cd^m(x_{2n-1}, x_{2n})d^m(x_{2n}, T_1x) + 8d(x_{2n}, T_1x)) \]
\[ \leq 8d^m(x_{2n}, T_1x) \]
for all \( n \) and so \( d(x, T_1x) \leq \frac{1}{c}d(x, T_1x) \), that is, \( d(T_1x, x) = 0 \). Since \( T_1x \) is a closed subset of \( X \), we conclude that \( x \in T_1x \). Now by using Lemma 2.1 and Theorem 2.2, we get \( \mathfrak{f}_{T_1} = \mathfrak{f}_{T_2} \) and \( \mathfrak{T}_1 = \mathfrak{T}_2 \).

References


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