About differential sandwich theorems using multiplier transformation and Ruscheweyh derivative

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Abstract: In this paper we obtain some subordination and superordination results for the operator \( IR_{m,n}^{\lambda,l} \) and we establish differential sandwich-type theorems. The operator \( IR_{m,n}^{\lambda,l} \) is defined as the Hadamard product of the multiplier transformation \( I(m,\lambda,l) \) and Ruscheweyh derivative \( R^n \).

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1 Introduction

Consider \( \mathcal{H}(U) \) the class of analytic function in the open unit disc of the complex plane \( U = \{ z \in \mathbb{C} : |z| < 1 \} \), \( \mathcal{H}(a,n) \) the subclass of \( \mathcal{H}(U) \) consisting of functions of the form \( f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \ldots \) and \( \mathcal{A}_n = \{ f \in \mathcal{H}(U) : f(z) = z + a_{n+1} z^{n+1} + \ldots, z \in U \} \) with \( \mathcal{A} = \mathcal{A}_1 \).

Next we remind the definition of differential subordination and superordination.

Let the functions \( f \) and \( g \) be analytic in \( U \). The function \( f \) is subordinate to \( g \), written \( f \prec g \), if there exists a Schwarz function \( w \), analytic in \( U \), with \( w(0) = 0 \) and \( |w(z)| < 1 \), for all \( z \in U \), such that \( f(z) = g(w(z)) \), for all \( z \in U \). In particular, if the function \( g \) is univalent in \( U \), the above subordination is equivalent to \( f(0) = g(0) \) and \( f(U) \subset g(U) \).

Let \( \psi : \mathbb{C}^3 \times U \to \mathbb{C} \) and \( h \) be an univalent function in \( U \). If \( p \) is analytic in \( U \) and satisfies the second order differential subordination

\[
\psi(p(z),zp'(z),z^2p''(z);z) < h(z), \quad \text{for } z \in U,
\] (1)

then \( p \) is called a solution of the differential subordination. The univalent function \( q \) is called a dominant of the solutions of the differential subordination, or more simply a dominant, if \( p \prec q \) for all \( p \) satisfying (1). A dominant \( \tilde{q} \) that satisfies \( \tilde{q} \prec q \) for all dominants \( q \) of (1) is said to be the best dominant of (1). The best dominant is unique up to a rotation of \( U \).

Let \( \psi : \mathbb{C}^2 \times U \to \mathbb{C} \) and \( h \) analytic in \( U \). If \( p \) and \( \psi(p(z),zp'(z),z^2p''(z);z) \) are univalent and if \( p \) satisfies the second order differential superordination

\[
h(z) \prec \psi(p(z),zp'(z),z^2p''(z);z), \quad z \in U,
\] (2)

then \( p \) is a solution of the differential superordination (2) (if \( f \) is subordinate to \( F \), then \( F \) is called to be superordinate to \( f \)). An analytic function \( q \) is called a subordinant if \( q \prec p \) for all
By simple computation we obtain the relation
\[ h(z) \prec \psi(p(z), z p'(z), z^2 p''(z); z) \Rightarrow q(z) \prec p(z). \]

For two functions \( f(z) = z + \sum_{j=2}^{\infty} a_j z^j \) and \( g(z) = z + \sum_{j=2}^{\infty} b_j z^j \) analytic in the open unit disc \( U \), the Hadamard product (or convolution) of \( f(z) \) and \( g(z) \), written as \((f \ast g)(z)\) is defined by
\[ (f \ast g)(z) = f(z) \ast g(z) = z + \sum_{j=2}^{\infty} a_j b_j z^j. \]

We need the following differential operators.

**Definition 1** [5] For \( f \in \mathcal{A} \), \( m \in \mathbb{N} \cup \{0\} \), \( \lambda, l \geq 0 \), the multiplier transformation \( I(m, \lambda, l) f(z) \) is defined by the following infinite series
\[ I(m, \lambda, l) f(z) := z + \sum_{j=2}^{\infty} \left( \frac{1 + \lambda (j-1) + l}{1 + l} \right)^m a_j z^j. \]

**Remark 1** We have
\[ (l+1) I(m+1, \lambda, l) f(z) = (l+1 - \lambda) I(m, \lambda, l) f(z) + \lambda z (I(m, \lambda, l) f(z))', \quad z \in U. \]

**Remark 2** For \( l = 0 \), \( \lambda \geq 0 \), the operator \( D^n_{\lambda} = I(m, \lambda, 0) \) was introduced and studied by Al-Oboudi, which reduced to the Sălăgean differential operator \( S^m = I(m, 1, 0) \) for \( \lambda = 1 \).

**Definition 2** (Ruscheweyh [8]) For \( f \in \mathcal{A} \) and \( n \in \mathbb{N} \), the Ruscheweyh derivative \( R^n \) is defined by \( R^n : \mathcal{A} \rightarrow \mathcal{A} \),
\[ R^0 f(z) = f(z) \]
\[ R^1 f(z) = zf'(z) \]
\[ \ldots \]
\[ (n+1) R^{n+1} f(z) = z (R^n f(z))' + n R^n f(z), \quad z \in U. \]

**Remark 3** If \( f \in \mathcal{A} \), \( f(z) = z + \sum_{j=2}^{\infty} a_j z^j \), then \( R^n f(z) = z + \sum_{j=2}^{\infty} \frac{(n+j-1)!}{n!(j-1)!} a_j z^j \) for \( z \in U \).

**Definition 3** ([2]) Let \( \lambda, l \geq 0 \) and \( m, n \in \mathbb{N} \). Denote by \( IR_{\lambda,l}^{m,n} : \mathcal{A} \rightarrow \mathcal{A} \) the operator given by the Hadamard product of the multiplier transformation \( I(m, \lambda, l) \) and the Ruscheweyh derivative \( R^n \),
\[ IR_{\lambda,l}^{m,n} f(z) = (I(m, \lambda, l) \ast R^n) f(z), \]
for any \( z \in U \) and each nonnegative integers \( m, n \).

**Remark 4** If \( f \in \mathcal{A} \) and \( f(z) = z + \sum_{j=2}^{\infty} a_j z^j \), then
\[ IR_{\lambda,l}^{m,n} f(z) = z + \sum_{j=2}^{\infty} \left( \frac{1 + \lambda (j-1) + l}{1 + l} \right)^m \frac{(n+j-1)!}{n!(j-1)!} a_j z^j, \quad z \in U. \]

By simple computation we obtain the relation
Proposition 1 [1] For \( m, n \in \mathbb{N} \) and \( \lambda, \ell \geq 0 \) we have
\[
(n + 1) \text{IR}_{\lambda, \ell}^{m,n+1} f(z) - n \text{IR}_{\lambda, \ell}^{m,n} f(z) = z \left( \text{IR}_{\lambda, \ell}^{m,n} f(z) \right)'.
\] (3)

We need the following

Definition 4 [7] Denote by \( Q \) the set of all functions \( f \) that are analytic and injective on \( U \setminus E(f) \), where \( E(f) = \{ \zeta \in \partial U : \lim_{z \to \zeta} f(z) = \infty \} \), and are such that \( f'(\zeta) \neq 0 \) for \( \zeta \in \partial U \setminus E(f) \).

Lemma 2 [7] Let the function \( q \) be univalent in the unit disc \( U \) and \( \theta \) and \( \phi \) be analytic in a domain \( D \) containing \( q(U) \) with \( \phi(w) \neq 0 \) when \( w \in q(U) \). Set \( Q(z) = zq'(z)\phi(q(z)) \) and \( h(z) = \theta(q(z)) + Q(z) \). Suppose that
1. \( Q \) is starlike univalent in \( U \) and
2. \( \text{Re} \left( \frac{zq'(z)}{Q(z)} \right) > 0 \) for \( z \in U \).

If \( p \) is analytic with \( p(0) = q(0) \), \( p(U) \subseteq D \) and
\[
\theta(p(z)) +zp'(z)\phi(p(z)) < \theta(q(z)) + zq'(z)\phi(q(z)),
\]
then \( p(z) < q(z) \) and \( q \) is the best dominant.

Lemma 3 [4] Let the function \( q \) be convex univalent in the open unit disc \( U \) and \( \nu \) and \( \phi \) be analytic in a domain \( D \) containing \( q(U) \). Suppose that
1. \( \text{Re} \left( \frac{\nu'(q(z))}{q'(q(z))} \right) > 0 \) for \( z \in U \) and
2. \( \psi(z) = zq'(z)\phi(q(z)) \) is starlike univalent in \( U \).

If \( p(z) \in \mathcal{H} [q(0), 1] \cap Q \), with \( p(U) \subseteq D \) and \( \nu(p(z)) + zp'(z)\phi(p(z)) \) is univalent in \( U \) and
\[
\nu(q(z)) + zq'(z)\phi(q(z)) < \nu(p(z)) + zp'(z)\phi(p(z)),
\]
then \( q(z) < p(z) \) and \( q \) is the best subordinate.

2 Main results

We intend to find sufficient conditions for certain normalized analytic functions \( f \) such that \( q_1(z) \prec \frac{z^{\delta} \text{IR}_{\lambda, \ell}^{m,n+1} f(z)}{(\text{IR}_{\lambda, \ell}^{m,n} f(z))^{1+\delta}} \prec q_2(z), \ z \in U, \ 0 < \delta \leq 1 \), where \( q_1 \) and \( q_2 \) are given univalent functions.

Theorem 4 Let \( \frac{z^{\delta} \text{IR}_{\lambda, \ell}^{m,n+1} f(z)}{(\text{IR}_{\lambda, \ell}^{m,n} f(z))^{1+\delta}} \in \mathcal{H}(U) \) and let the function \( q(z) \) be analytic and univalent in \( U \) such that \( q(z) \neq 0 \), for all \( z \in U \). Suppose that \( \frac{zq'(z)}{q(z)} \) is starlike univalent in \( U \). Let
\[
\text{Re} \left( \frac{\xi}{\beta} q(z) + \frac{2\mu}{\beta} q^2(z) + 1 + zq''(z) - zq'(z) \right) > 0,
\] (4)
for \( \alpha, \xi, \beta, \mu \in \mathbb{C}, \ \beta \neq 0, \ z \in U \) and
\[
\psi_{\lambda, \ell}^{m,n} (\alpha, \xi, \mu, \beta; z) := \alpha + \beta \delta \left( n + 1 \right) + \beta \left( n + 1 \right) \frac{\text{IR}_{\lambda, \ell}^{m,n+2} f(z)}{\text{IR}_{\lambda, \ell}^{m,n+1} f(z)},
\] (5)
By an application of Lemma 2, we have

\[
\beta (1 + \delta) (n + 1) \frac{IR_{\lambda \mu}^{m,n+1} f(z)}{IR_{\lambda \mu}^{m,n} f(z)} + \xi \frac{z^\delta IR_{\lambda \mu}^{m,n+1} f(z)}{(IR_{\lambda \mu}^{m,n} f(z))^{1+\delta}} + \mu z^{2\delta} \frac{(IR_{\lambda \mu}^{m,n+1} f(z))^2}{(IR_{\lambda \mu}^{m,n} f(z))^{2+2\delta}}.
\]

If \( q \) satisfies the following subordination

\[
\psi_{\lambda \mu}^{m,n} (\alpha, \beta, \mu; z) \prec \alpha + \xi q(z) + \mu (q(z))^2 + \beta \frac{z q'(z)}{q(z)},
\]

for \( \alpha, \xi, \beta, \mu \in \mathbb{C}, \beta \neq 0 \), then

\[
\frac{z^\delta IR_{\lambda \mu}^{m,n+1} f(z)}{(IR_{\lambda \mu}^{m,n} f(z))^{1+\delta}} \prec q(z),
\]

and \( q \) is the best dominant.

**Proof.** Consider \( p(z) := \frac{z^\delta IR_{\lambda \mu}^{m,n+1} f(z)}{(IR_{\lambda \mu}^{m,n} f(z))^{1+\delta}}, z \in U, z \neq 0, f \in A \). We have \( p'(z) = \delta (1 + n) \frac{z^{\delta - 1} IR_{\lambda \mu}^{m,n+1} f(z)}{(IR_{\lambda \mu}^{m,n} f(z))^{1+\delta}} + (n + 1) \frac{z^{\delta - 1} IR_{\lambda \mu}^{m,n+1} f(z)}{(IR_{\lambda \mu}^{m,n} f(z))^{1+\delta}} - (1 + \delta) (n + 1) \frac{z^{\delta - 1} IR_{\lambda \mu}^{m,n+1} f(z)}{(IR_{\lambda \mu}^{m,n} f(z))^{1+\delta}} \).

By using the identity (3), we obtain

\[
\frac{zp'(z)}{p(z)} = \delta (1 + n) + (n + 1) \frac{IR_{\lambda \mu}^{m,n+2} f(z)}{IR_{\lambda \mu}^{m,n+1} f(z)} - (1 + \delta) (n + 1) \frac{z^{\delta - 1} IR_{\lambda \mu}^{m,n+1} f(z)}{IR_{\lambda \mu}^{m,n} f(z)}.
\]

By setting \( \theta (w) := \alpha + \xi w + \mu w^2 \) and \( \phi (w) := \frac{\beta}{w} \), it can be easily verified that \( \theta \) is analytic in \( \mathbb{C} \), \( \phi \) is analytic in \( \mathbb{C} \setminus \{0\} \) and that \( \phi(w) \neq 0, w \in \mathbb{C} \setminus \{0\} \).

Also, by letting \( Q(z) = zq'(z) \phi(q(z)) = \beta zq'(z) \) and \( h(z) = \theta(q(z)) + Q(z) = \alpha + \xi q(z) + \mu (q(z))^2 + \beta \frac{z q'(z)}{q(z)} \), we find that \( Q(z) \) is starlike univalent in \( U \).

We get \( h'(z) = \xi q'(z) + 2\mu q(z) q'(z) + \beta \frac{q'(z)}{q(z)} + \beta z q'(z) - \beta z \left( \frac{q'(z)}{q(z)} \right)^2 \) and \( \frac{zh'(z)}{q(z)} = \frac{\xi q(z) + 2\mu q(z) q'(z) + 1 + z \frac{q'(z)}{q(z)} - \frac{z q'(z)}{q(z)}}{q(z)} \).

We deduce that \( \Re \left( \frac{zh'(z)}{q(z)} \right) = \Re \left( \frac{\xi q(z) + 2\mu q(z) q'(z) + 1 + z \frac{q'(z)}{q(z)} - \frac{z q'(z)}{q(z)}}{q(z)} \right) > 0 \).

By using (8), we obtain

\[
\alpha + \xi p(z) + \mu (p(z))^2 + \beta \frac{z q'(z)}{p(z)} = \alpha + \xi \delta (n + 1) + \beta (n + 1) \frac{IR_{\lambda \mu}^{m,n+2} f(z)}{IR_{\lambda \mu}^{m,n+1} f(z)} - \beta (1 + \delta) (n + 1) \frac{IR_{\lambda \mu}^{m,n+1} f(z)}{IR_{\lambda \mu}^{m,n} f(z)} + \xi \frac{z^\delta IR_{\lambda \mu}^{m,n+1} f(z)}{(IR_{\lambda \mu}^{m,n} f(z))^{1+\delta}} + \mu \frac{z^{2\delta} (IR_{\lambda \mu}^{m,n+1} f(z))^2}{(IR_{\lambda \mu}^{m,n} f(z))^{2+2\delta}}.
\]

By using (6), we have \( \alpha + \xi p(z) + \mu (p(z))^2 + \beta \frac{z q'(z)}{p(z)} \prec \alpha + \xi q(z) + \mu (q(z))^2 + \beta \frac{z q'(z)}{q(z)} \).

By an application of Lemma 2, we have \( p(z) \prec q(z), z \in U, \) i.e. \( \frac{z^\delta IR_{\lambda \mu}^{m,n+1} f(z)}{(IR_{\lambda \mu}^{m,n} f(z))^{1+\delta}} \prec q(z), z \in U \) and \( q \) is the best dominant. \( \square \)

**Corollary 5** Let \( m,n \in \mathbb{N}, \lambda,l \geq 0 \). Assume that (4) holds. If \( f \in A \) and

\[
\psi_{\lambda \mu}^{m,n} (\alpha, \beta, \mu; z) \prec \alpha + \xi \left( \frac{1 + Az}{1 + Bz} \right)^2 + \beta \frac{(1 + Az)(1 + Bz)}{(1 + Az)(1 + Bz)}.
\]
for \(\alpha, \beta, \mu, \xi \in \mathbb{C}, \beta \neq 0, -1 \leq B < A \leq 1\), where \(\psi^{m,n}_{\lambda,l}\) is defined in (5), then

\[
\frac{z^\delta I R^{m,n+1}_{\lambda,l} f(z)}{I R^{m,n}_{\lambda,l} f(z)} \prec \frac{1 + A z}{1 + B z},
\]

and \(\frac{1 + A z}{1 + B z}\) is the best dominant.

**Proof.** For \(q(z) = \frac{1 + A z}{1 + B z}\), \(-1 \leq B < A \leq 1\) in Theorem 4 we get the corollary. \(\blacksquare\)

**Corollary 6** Let \(m, n \in \mathbb{N}, \lambda, l \geq 0\). Assume that (4) holds. If \(f \in \mathcal{A}\) and

\[
\psi^{m,n}_{\lambda,l}(\alpha, \beta, \mu; z) \prec \alpha + \xi \left(\frac{1 + z}{1 - z}\right)^\gamma + \mu \left(\frac{1 + z}{1 - z}\right)^{2\gamma} + \frac{2\beta\gamma z}{1 - z^2},
\]

for \(\alpha, \beta, \mu, \xi \in \mathbb{C}, 0 < \gamma \leq 1, \beta \neq 0\), where \(\psi^{m,n}_{\lambda,l}\) is defined in (5), then

\[
\frac{z^\delta I R^{m,n+1}_{\lambda,l} f(z)}{I R^{m,n}_{\lambda,l} f(z)} \prec \left(\frac{1 + z}{1 - z}\right)^\gamma,
\]

and \(\left(\frac{1 + z}{1 - z}\right)^\gamma\) is the best dominant.

**Proof.** Corollary follows by using Theorem 4 for \(q(z) = \left(\frac{1 + z}{1 - z}\right)^\gamma, 0 < \gamma \leq 1\). \(\blacksquare\)

**Theorem 7** Let \(q\) be analytic and univalent in \(U\) such that \(q(z) \neq 0\) and \(\frac{zq'(z)}{q(z)}\) be starlike univalent in \(U\). Assume that

\[
Re\left(\frac{\xi}{\beta} q(z) q'(z) + \frac{2\mu}{\beta} q^2(z) q'(z)\right) > 0, \text{ for } \xi, \beta, \mu \in \mathbb{C}, \beta \neq 0.
\]

If \(f \in \mathcal{A}\), \(\frac{z^\delta I R^{m,n+1}_{\lambda,l} f(z)}{(I R^{m,n}_{\lambda,l} f(z))} \in \mathcal{H}[q(0), 1] \cap Q\) and \(\psi^{m,n}_{\lambda,l}(\alpha, \beta, \mu; z)\) is univalent in \(U\), where \(\psi^{m,n}_{\lambda,l}(\alpha, \beta, \mu; z)\) is as defined in (5), then

\[
\alpha + \xi q(z) + \mu (q(z))^2 + \frac{\beta q'(z)}{q(z)} \prec \psi^{m,n}_{\lambda,l}(\alpha, \beta, \mu; z)
\]

implies

\[
q(z) \prec \frac{z^\delta I R^{m,n+1}_{\lambda,l} f(z)}{(I R^{m,n}_{\lambda,l} f(z))}^{1+\delta}, \quad z \in U,
\]

and \(q\) is the best subordinant.

**Proof.** Consider \(p(z) := \frac{z^\delta I R^{m,n+1}_{\lambda,l} f(z)}{(I R^{m,n}_{\lambda,l} f(z))^{1+\delta}}, z \in U, z \neq 0, f \in \mathcal{A}\).

By setting \(\nu(w) := \alpha + \xi w + \mu w^2\) and \(\phi(w) := \frac{\beta}{w}\) it can be easily verified that \(\nu\) is analytic in \(\mathbb{C}\), \(\phi\) is analytic in \(\mathbb{C} \setminus \{0\}\) and that \(\phi(w) \neq 0, w \in \mathbb{C} \setminus \{0\}\).

Since \(\frac{\nu'(q(z))}{\phi(q(z))} = \frac{q'(z)q(z)(\xi + 2\mu q(z))}{\beta}\), it follows that \(Re\left(\frac{\nu'(q(z))}{\phi(q(z))}\right) = Re\left(\frac{\xi}{\beta} q(z) q'(z) + \frac{2\mu}{\beta} q^2(z) q'(z)\right) > 0, \text{ for } \alpha, \beta, \mu \in \mathbb{C}, \mu \neq 0.\)
By using (8) and (10) we obtain
\[ \alpha + \xi q(z) + \mu (q(z))^2 + \frac{\beta z q'(z)}{q(z)} \prec \alpha + \xi p(z) + \mu (p(z))^2 + \frac{\beta z p'(z)}{p(z)}. \]

Applying Lemma 3, we get
\[ q(z) \prec p(z) = z^{\delta \IR^{m,n}_{\lambda,l} f(z)} \prec z^{\delta \IR^{m,n}_{\lambda,l} f(z)} \]
and \( q \) is the best subordinant. \( \blacksquare \)

**Corollary 8** Let \( m,n \in \mathbb{N}, \lambda, l \geq 0 \). Assume that (9) holds. If \( f \in A, \frac{z^\delta \IR^{m,n+1}_{\lambda,l} f(z)}{(IR^{m,n}_{\lambda,l} f(z))^{1+\delta}} \in \mathcal{H}[q(0), 1] \cap Q \) and
\[ \alpha + \xi \frac{1+Az}{1+Bz} + \mu \left( \frac{1+A z}{1+B z} \right)^2 + \frac{\beta (A-B) z}{(1+A z)(1+B z)} \prec \psi_{\lambda,l}^{m,n}(\alpha, \beta, \mu; z), \]
for \( \alpha, \beta, \xi, \mu \in \mathbb{C}, \beta \neq 0, -1 \leq B < A \leq 1 \), where \( \psi_{\lambda,l}^{m,n} \) is defined in (5), then
\[ \frac{1+Az}{1+Bz} \prec \frac{z^\delta \IR^{m,n+1}_{\lambda,l} f(z)}{(IR^{m,n}_{\lambda,l} f(z))^{1+\delta}}, \]
and \( \frac{1+Az}{1+Bz} \) is the best subordinant.

**Proof.** For \( q(z) = \frac{1+Az}{1+Bz}, -1 \leq B < A \leq 1 \) in Theorem 7 we get the corollary. \( \blacksquare \)

**Corollary 9** Let \( m,n \in \mathbb{N}, \lambda, l \geq 0 \). Assume that (9) holds. If \( f \in A, \frac{z^\delta \IR^{m,n+1}_{\lambda,l} f(z)}{(IR^{m,n}_{\lambda,l} f(z))^{1+\delta}} \in \mathcal{H}[q(0), 1] \cap Q \) and
\[ \alpha + \xi \left( \frac{1+z}{1-z} \right)^\gamma + \mu \left( \frac{1+z}{1-z} \right)^2 + \frac{2\beta \gamma z}{1-z^2} \prec \psi_{\lambda,l}^{m,n}(\alpha, \beta, \mu; z), \]
for \( \alpha, \beta, \mu, \xi \in \mathbb{C}, \beta \neq 0, 0 < \gamma \leq 1 \), where \( \psi_{\lambda,l}^{m,n} \) is defined in (5), then
\[ \left( \frac{1+z}{1-z} \right)^\gamma \prec \frac{z^\delta \IR^{m,n+1}_{\lambda,l} f(z)}{(IR^{m,n}_{\lambda,l} f(z))^{1+\delta}}, \]
and \( \left( \frac{1+z}{1-z} \right)^\gamma \) is the best subordinant.

**Proof.** For \( q(z) = \left( \frac{1+z}{1-z} \right)^\gamma, 0 < \gamma \leq 1 \) in Theorem 7 we get the corollary. \( \blacksquare \)

Combining Theorem 4 and Theorem 7, we state the following sandwich theorem.

**Theorem 10** Let \( q_1 \) and \( q_2 \) be analytic and univalent in \( U \) such that \( q_1(z) \neq 0 \) and \( q_2(z) \neq 0 \), for all \( z \in U \), with \( \frac{z q_1'(z)}{q_1(z)} \) and \( \frac{z q_2'(z)}{q_2(z)} \) being starlike univalent. Suppose that \( q_1 \) satisfies (4) and \( q_2 \)
satisfies (9). If \( f \in \mathcal{A}, \frac{z^{d_{\mathcal{IR}_{\lambda,l}^{m,n+1}}} f(z)}{(IR_{\lambda,l}^{m,n} f(z))^{1+d}} \in \mathcal{H} [q(0), 1] \cap Q \) and \( \psi_{\lambda,l}^{m,n} (\alpha, \beta, \mu; z) \) is as defined in (5) univalent in \( U \), then

\[
\alpha + \xi q_1(z) + \mu (q_1(z))^2 + \frac{\beta q_1'(z)}{q_1(z)} < \psi_{\lambda,l}^{m,n} (\alpha, \beta, \mu; z) < \alpha + \xi q_2(z) + \mu (q_2(z))^2 + \frac{\beta q_2'(z)}{q_2(z)}
\]

for \( \alpha, \beta, \mu, \xi \in \mathbb{C}, \beta \neq 0 \), implies

\[
q_1(z) < \frac{z^{d_{\mathcal{IR}_{\lambda,l}^{m,n+1}}} f(z)}{(IR_{\lambda,l}^{m,n} f(z))^{1+d}} < q_2(z),
\]

and \( q_1 \) and \( q_2 \) are respectively the best subordinant and the best dominant.

For \( q_1(z) = \frac{1+A_1 z}{1+B_1 z}, q_2(z) = \frac{1+A_2 z}{1+B_2 z} \), where \(-1 \leq B_2 < B_1 < A_1 < A_2 \leq 1\), we have the following corollary.

**Corollary 11** Let \( m, n \in \mathbb{N}, \lambda, l \geq 0 \). Assume that (4) and (9) hold. If \( f \in \mathcal{A}, \frac{z^{d_{\mathcal{IR}_{\lambda,l}^{m,n+1}}} f(z)}{(IR_{\lambda,l}^{m,n} f(z))^{1+d}} \in \mathcal{H} [q(0), 1] \cap Q \) and

\[
\alpha + \xi \left( \frac{1+A_1 z}{1+B_1 z} \right)^{\gamma_1} + \mu \left( \frac{1+A_1 z}{1+B_1 z} \right)^{2\gamma_1} + \frac{\beta (A_1 - B_1) z}{(1+A_1 z)(1+B_1 z)} < \psi_{\lambda,l}^{m,n} (\alpha, \beta, \mu; z)
\]

\[
< \alpha + \xi \left( \frac{1+A_2 z}{1+B_2 z} \right)^{\gamma_2} + \mu \left( \frac{1+A_2 z}{1+B_2 z} \right)^{2\gamma_2} + \frac{\beta (A_2 - B_2) z}{(1+A_2 z)(1+B_2 z)},
\]

for \( \alpha, \beta, \mu, \xi \in \mathbb{C}, \beta \neq 0, -1 \leq B_2 \leq B_1 < A_1 \leq A_2 \leq 1 \), where \( \psi_{\lambda,l}^{m,n} \) is defined in (5), then

\[
\frac{1+A_1 z}{1+B_1 z} < \frac{z^{d_{\mathcal{IR}_{\lambda,l}^{m,n+1}}} f(z)}{(IR_{\lambda,l}^{m,n} f(z))^{1+d}} < \frac{1+A_2 z}{1+B_2 z},
\]

hence \( \frac{1+A_1 z}{1+B_1 z} \) and \( \frac{1+A_2 z}{1+B_2 z} \) are the best subordinant and the best dominant, respectively.

For \( q_1(z) = \left( \frac{1+z}{1-z} \right)^{\gamma_1}, q_2(z) = \left( \frac{1+z}{1-z} \right)^{\gamma_2} \), where \( 0 < \gamma_1 < \gamma_2 \leq 1 \), we have the following corollary.

**Corollary 12** Let \( m, n \in \mathbb{N}, \lambda, l \geq 0 \). Assume that (4) and (9) hold. If \( f \in \mathcal{A}, \frac{z^{d_{\mathcal{IR}_{\lambda,l}^{m,n+1}}} f(z)}{(IR_{\lambda,l}^{m,n} f(z))^{1+d}} \in \mathcal{H} [q(0), 1] \cap Q \) and

\[
\alpha + \xi \left( \frac{1+z}{1-z} \right)^{\gamma_1} + \mu \left( \frac{1+z}{1-z} \right)^{2\gamma_1} + \frac{\beta \gamma_1 z}{1-z^2} < \psi_{\lambda,l}^{m,n} (\alpha, \beta, \mu; z)
\]

\[
< \alpha + \xi \left( \frac{1+z}{1-z} \right)^{\gamma_2} + \mu \left( \frac{1+z}{1-z} \right)^{2\gamma_2} + \frac{\beta \gamma_2 z}{1-z^2},
\]

for \( \alpha, \beta, \mu, \xi \in \mathbb{C}, \beta \neq 0, 0 < \gamma_1 < \gamma_2 \leq 1 \), where \( \psi_{\lambda,l}^{m,n} \) is defined in (5), then

\[
\left( \frac{1+z}{1-z} \right)^{\gamma_1} < \frac{z^{d_{\mathcal{IR}_{\lambda,l}^{m,n+1}}} f(z)}{(IR_{\lambda,l}^{m,n} f(z))^{1+d}} < \left( \frac{1+z}{1-z} \right)^{\gamma_2},
\]

hence \( \left( \frac{1+z}{1-z} \right)^{\gamma_1} \) and \( \left( \frac{1+z}{1-z} \right)^{\gamma_2} \) are the best subordinant and the best dominant, respectively.
Changing the functions $\theta$ and $\phi$ we obtain the following results.

**Theorem 13** Let $\frac{z^{\delta}IR_{m,n}^{m,n+1}f(z)}{(IR_{\lambda,l}^{m,n}f(z))^{1+\delta}} \in H(U)$, $f \in A$, $z \in U$, $m, n \in \mathbb{N}$, $\lambda, l \geq 0$ and let the function $q(z)$ be convex and univalent in $U$ such that $q(0) = 1$, $z \in U$. Assume that

$$Re\left(\frac{\alpha + \beta}{\beta} + z\frac{q''(z)}{q'(z)}\right) > 0,$$  \hspace{1cm} (12)

for $\alpha, \beta \in \mathbb{C}$, $\beta \neq 0$, $z \in U$, and

$$\psi_{\lambda,l}^{m,n}(\alpha, \beta; z) := (\alpha + \beta \delta (n + 1)) \frac{z^{\delta}IR_{\lambda,l}^{m,n+1}f(z)}{(IR_{\lambda,l}^{m,n}f(z))^{1+\delta}} + \beta (n + 1) \frac{z^{\delta}IR_{\lambda,l}^{m,n+2}f(z)}{(IR_{\lambda,l}^{m,n}f(z))^{1+\delta}} - \beta (1 + \delta) (n + 1) \frac{z^{\delta}(IR_{\lambda,l}^{m,n+1}f(z))^{2}}{(IR_{\lambda,l}^{m,n}f(z))^{2+\delta}}.$$  \hspace{1cm} (13)

If $q$ satisfies the following subordination

$$\psi_{\lambda,l}^{m,n}(\alpha, \beta; z) < \alpha q(z) + \beta zq'(z),$$  \hspace{1cm} (14)

for $\alpha, \beta \in \mathbb{C}$, $\beta \neq 0$, $z \in U$, then

$$\frac{z^{\delta}IR_{\lambda,l}^{m,n+1}f(z)}{(IR_{\lambda,l}^{m,n}f(z))^{1+\delta}} < q(z), \quad z \in U,$$  \hspace{1cm} (15)

and $q$ is the best dominant.

**Proof.** Consider $p(z) := \frac{z^{\delta}IR_{m,n}^{m,n+1}f(z)}{(IR_{\lambda,l}^{m,n}f(z))^{1+\delta}}, \quad z \in U$, $z \neq 0$, $f \in A$. The function $p$ is analytic in $U$ and $p(0) = 1$

We have $p'(z) = \delta (1 + n) \frac{z^{\delta-1}IR_{m,n}^{m,n+1}f(z)}{(IR_{\lambda,l}^{m,n}f(z))^{1+\delta}} + (n + 1) \frac{z^{\delta-1}IR_{m,n}^{m,n+2}f(z)}{(IR_{\lambda,l}^{m,n}f(z))^{1+\delta}} - (1 + \delta) (n + 1) \frac{z^{\delta-1}(IR_{\lambda,l}^{m,n+1}f(z))^{2}}{(IR_{\lambda,l}^{m,n}f(z))^{2+\delta}}.$

By using the identity (3), we obtain

$$zp'(z) = \delta (1 + n) \frac{z^{\delta}IR_{\lambda}^{m,n}f(z)}{(IR_{\lambda}^{m,n}f(z))^{1+\delta}} + (n + 1) \frac{z^{\delta}IR_{\lambda}^{m,n+2}f(z)}{(IR_{\lambda}^{m,n}f(z))^{1+\delta}} - (1 + \delta) (n + 1) \frac{z^{\delta}(IR_{\lambda}^{m,n+1}f(z))^{2}}{(IR_{\lambda}^{m,n}f(z))^{2+\delta}}.$$  \hspace{1cm} (16)

By setting $\theta(w) := \alpha w$ and $\phi(w) := \beta$, it can be easily verified that $\theta$ is analytic in $\mathbb{C} \setminus \{0\}$ and that $\phi(w) \neq 0$, $w \in \mathbb{C} \setminus \{0\}$.

Also, by letting $Q(z) = zq'(z) \phi(q(z)) = \beta zq'(z)$, we find that $Q(z)$ is starlike univalent in $U$.

Let $h(z) = \theta(q(z)) + Q(z) = \alpha q(z) + \beta zq'(z)$.

We have $Re\left(\frac{h'(z)}{Q'(z)}\right) = Re\left(\frac{\alpha + \beta}{\beta} + z\frac{q''(z)}{q'(z)}\right) > 0$.

By using (16), we obtain $\alpha p(z) + \beta zq'(z) = (\alpha + \beta \delta (n + 1)) \frac{z^{\delta}IR_{\lambda}^{m,n}f(z)}{(IR_{\lambda}^{m,n}f(z))^{1+\delta}} + \beta (n + 1) \frac{z^{\delta}(IR_{\lambda}^{m,n+1}f(z))^{2}}{(IR_{\lambda}^{m,n}f(z))^{2+\delta}}.$

$$\frac{z^{\delta}IR_{\lambda}^{m,n+2}f(z)}{(IR_{\lambda}^{m,n}f(z))^{1+\delta}} - \beta (1 + \delta) (n + 1) \frac{z^{\delta}(IR_{\lambda}^{m,n+1}f(z))^{2}}{(IR_{\lambda}^{m,n}f(z))^{2+\delta}}.$$
By using (14), we have \( \alpha p(z) + \beta z p'(z) < \alpha q(z) + \beta z q'(z) \).

Applying Lemma 2, we get \( p(z) < q(z) \), \( z \in U \), i.e. \( \frac{z^i IR_{\lambda,l}^{m,n+1} f(z)}{(IR_{\lambda,l}^{m,n} f(z))^{1+\delta}} < q(z) \), \( z \in U \), and \( q \) is the best dominant. 

**Corollary 14** Let \( q(z) = \frac{1+A_z}{1+B_z} \), \( z \in U \), \( -1 \leq B < A \leq 1 \), \( m, n \in \mathbb{N} \), \( \lambda, l \geq 0 \). Assume that (12) holds. If \( f \in \mathcal{A} \) and

\[
\psi_{\lambda,l}^{m,n} (\alpha, \beta; z) < \alpha \frac{1 + A_z}{1 + B_z} + \beta \frac{A - B}{(1 + B_z)^2},
\]

for \( \alpha, \beta \in \mathbb{C} \), \( \beta \neq 0 \), \( -1 \leq B < A \leq 1 \), where \( \psi_{\lambda,l}^{m,n} \) is defined in (13), then

\[
\frac{z^i IR_{\lambda,l}^{m,n+1} f(z)}{(IR_{\lambda,l}^{m,n} f(z))^{1+\delta}} < \frac{1 + A_z}{1 + B_z},
\]

and \( \frac{1+A_z}{1+B_z} \) is the best dominant.

**Proof.** For \( q(z) = \frac{1+A_z}{1+B_z} \), \( -1 \leq B < A \leq 1 \), in Theorem 13 we get the corollary. 

**Corollary 15** Let \( q(z) = \left( \frac{1+z}{1-z} \right)^\gamma \), \( m, n \in \mathbb{N} \), \( \lambda, l \geq 0 \). Assume that (12) holds. If \( f \in \mathcal{A} \) and

\[
\psi_{\lambda,l}^{m,n} (\alpha, \beta; z) < \alpha \left( \frac{1 + z}{1 - z} \right)^\gamma + \frac{2 \beta \gamma z}{1 - z^2} \left( \frac{1 + z}{1 - z} \right)^\gamma,
\]

for \( \alpha, \beta \in \mathbb{C} \), \( 0 < \gamma \leq 1 \), \( \beta \neq 0 \), where \( \psi_{\lambda,l}^{m,n} \) is defined in (13), then

\[
\frac{z^i IR_{\lambda,l}^{m,n+1} f(z)}{(IR_{\lambda,l}^{m,n} f(z))^{1+\delta}} < \left( \frac{1 + z}{1 - z} \right)^\gamma,
\]

and \( \left( \frac{1+z}{1-z} \right)^\gamma \) is the best dominant.

**Proof.** Corollary follows by using Theorem 13 for \( q(z) = \left( \frac{1+z}{1-z} \right)^\gamma \), \( 0 < \gamma \leq 1 \). 

**Theorem 16** Let \( q \) be convex and univalent in \( U \) such that \( q(0) = 1 \). Assume that

\[
\text{Re} \left( \frac{\alpha}{\beta} q'(z) \right) > 0, \text{ for } \alpha, \beta \in \mathbb{C}, \beta \neq 0.
\]  \hspace{1cm} (17)

If \( f \in \mathcal{A} \), \( \frac{z^i IR_{\lambda,l}^{m,n+1} f(z)}{(IR_{\lambda,l}^{m,n} f(z))^{1+\delta}} \in \mathcal{H} [q(0), 1] \cap Q \) and \( \psi_{\lambda,l}^{m,n} (\alpha, \beta; z) \) is univalent in \( U \), where \( \psi_{\lambda,l}^{m,n} (\alpha, \beta; z) \) is as defined in (13), then

\[
\alpha q(z) + \beta z q'(z) < \psi_{\lambda,l}^{m,n} (\alpha, \beta; z)
\]  \hspace{1cm} (18)

implies

\[
q(z) < \frac{z^i IR_{\lambda,l}^{m,n+1} f(z)}{(IR_{\lambda,l}^{m,n} f(z))^{1+\delta}}, \text{ for } \delta \in \mathbb{C}, \delta \neq 0, \text{ and } z \in U,
\]  \hspace{1cm} (19)

and \( q \) is the best subordinant.
Now, by using (18) we obtain

Combining Theorem 13 and Theorem 16, we state the following sandwich theorem.

Proof. Consider \( p(z) := \frac{z^2 IR_{\lambda,l,m,n}^{m+1} f(z)}{(IR_{\lambda,l,m,n}^{m,n} f(z))^{1+\sigma}} \), \( z \in U, \ z \neq 0, \ f \in \mathcal{A} \). The function \( p \) is analytic in \( U \) and \( p(0) = 1 \).

By setting \( \nu(w) := \alpha w \) and \( \phi(w) := \beta \) it can be easily verified that \( \nu \) is analytic in \( \mathbb{C} \), \( \phi \) is analytic in \( \mathbb{C}\setminus\{0\} \) and that \( \phi(w) \neq 0, \ w \in \mathbb{C}\setminus\{0\} \).

Since \( \nu(q(z)) = \beta q'(z) \), it follows that \( Re\left(\frac{\nu'(q(z))}{\phi(q(z))}\right) = Re\left(\frac{2}{\beta} q'(z)\right) > 0 \), for \( \alpha, \beta \in \mathbb{C}, \beta \neq 0 \).

Now, by using (18) we obtain

\[
\alpha q(z) + \beta zq'(z) \prec \alpha p(z) + \beta zp'(z), \quad z \in U.
\]

Applying Lemma 3, we get

\[
q(z) \prec p(z) = \frac{z^2 IR_{\lambda,l,m,n}^{m+1} f(z)}{(IR_{\lambda,l,m,n}^{m,n} f(z))^{1+\sigma}}, \quad z \in U,
\]

and \( q \) is the best subordinant. ■

Corollary 17 Let \( q(z) = \frac{1+A z}{1+ B z}, -1 \leq B < A \leq 1, \ z \in U, \ m, n \in \mathbb{N}, \ \lambda, l \geq 0 \). Assume that (17) holds. If \( f \in \mathcal{A} \), \( \frac{z^2 IR_{\lambda,l,m,n}^{m+1} f(z)}{(IR_{\lambda,l,m,n}^{m,n} f(z))^{1+\sigma}} \in \mathcal{H}[q(0), 1] \cap Q \), and

\[
\alpha \frac{1 + Az}{1 + Bz} + \beta \frac{A - B}{1 + Bz} z \prec \psi_{\lambda,l,m,n}(\alpha, \beta; z),
\]

for \( \alpha, \beta \in \mathbb{C}, \beta \neq 0, \ -1 \leq B < A \leq 1 \), where \( \psi_{\lambda,l,m,n} \) is defined in (13), then

\[
\frac{1 + Az}{1 + Bz} \prec \frac{z^2 IR_{\lambda,l,m,n}^{m+1} f(z)}{(IR_{\lambda,l,m,n}^{m,n} f(z))^{1+\sigma}},
\]

and \( \frac{1+A z}{1+ B z} \) is the best subordinant.

Proof. For \( q(z) = \frac{1+A z}{1+ B z}, -1 \leq B < A \leq 1 \), in Theorem 16 we get the corollary. ■

Corollary 18 Let \( q(z) = \left(\frac{1+z}{1-z}\right)^{\gamma}, m, n \in \mathbb{N}, \ \lambda, l \geq 0 \). Assume that (17) holds. If \( f \in \mathcal{A} \), \( \frac{z^2 IR_{\lambda,l,m,n}^{m+1} f(z)}{(IR_{\lambda,l,m,n}^{m,n} f(z))^{1+\sigma}} \in \mathcal{H}[q(0), 1] \cap Q \) and

\[
\alpha \left(\frac{1+z}{1-z}\right)^{\gamma} + \frac{2 \beta \gamma z}{1-z} \left(\frac{1+z}{1-z}\right)^{\gamma} \prec \psi_{\lambda,l,m,n}(\alpha, \beta; z),
\]

for \( \alpha, \beta \in \mathbb{C}, \ 0 < \gamma \leq 1, \ \beta \neq 0 \), where \( \psi_{\lambda,l,m,n} \) is defined in (13), then

\[
\left(\frac{1+z}{1-z}\right)^{\gamma} \prec \frac{z^2 IR_{\lambda,l,m,n}^{m+1} f(z)}{(IR_{\lambda,l,m,n}^{m,n} f(z))^{1+\sigma}},
\]

and \( \left(\frac{1+z}{1-z}\right)^{\gamma} \) is the best subordinant.

Proof. Corollary follows by using Theorem 16 for \( q(z) = \left(\frac{1+z}{1-z}\right)^{\gamma}, \ 0 < \gamma \leq 1 \). ■

Combining Theorem 13 and Theorem 16, we state the following sandwich theorem.
Corollary 20 Let $q_1$ and $q_2$ be convex and univalent in $U$ such that $q_1(z) \neq 0$ and $q_2(z) \neq 0$, for all $z \in U$. Suppose that $q_1$ satisfies (12) and $q_2$ satisfies (17). If $f \in \mathcal{A}$, \(\frac{z^2 IR_{\lambda,l}^{m,n+1} f(z)}{(IR_{\lambda,l}^{m,n} f(z))^{1+\delta}} \in \mathcal{H}[q(0),1] \cap Q\), and $\psi_{\lambda,l}^{m,n} (\alpha, \beta; z)$ is as defined in (13) univalent in $U$, then
\[
\alpha q_1(z) + \beta z q_1'(z) < \psi_{\lambda,l}^{m,n} (\alpha, \beta; z) < \alpha q_2(z) + \beta z q_2'(z),
\]
for $\alpha, \beta \in \mathbb{C}$, $\beta \neq 0$, implies
\[
q_1(z) < \frac{z^2 IR_{\lambda,l}^{m,n+1} f(z)}{(IR_{\lambda,l}^{m,n} f(z))^{1+\delta}} < q_2(z), \quad z \in U,
\]
and $q_1$ and $q_2$ are respectively the best subordinant and the best dominant.

For $q_1(z) = \frac{1 + A_1 z}{1 + B_1 z}$, $q_2(z) = \frac{1 + A_2 z}{1 + B_2 z}$, where $-1 \leq B_2 < B_1 < A_1 < A_2 \leq 1$, we have the following corollary.

Corollary 21 Let $m,n \in \mathbb{N}$, $\lambda,l \geq 0$. Assume that (12) and (17) hold for $q_1(z) = \frac{1 + A_1 z}{1 + B_1 z}$ and $q_2(z) = \frac{1 + A_2 z}{1 + B_2 z}$, respectively. If $f \in \mathcal{A}$, \(\frac{z^2 IR_{\lambda,l}^{m,n+1} f(z)}{(IR_{\lambda,l}^{m,n} f(z))^{1+\delta}} \in \mathcal{H}[q(0),1] \cap Q\) and
\[
\alpha \frac{1 + A_1 z}{1 + B_1 z} + \beta \frac{(A_1 - B_1) z}{(1 + B_1 z)^2} < \psi_{\lambda,l}^{m,n} (\alpha, \beta; z)
\]
\[
< \alpha \frac{1 + A_2 z}{1 + B_2 z} + \beta \frac{(A_2 - B_2) z}{(1 + B_2 z)^2}, \quad z \in U,
\]
for $\alpha, \beta \in \mathbb{C}$, $\beta \neq 0$, $-1 \leq B_2 < B_1 < A_1 < A_2 \leq 1$, where $\psi_{\lambda,l}^{m,n}$ is defined in (5), then
\[
\frac{1 + A_1 z}{1 + B_1 z} < \frac{z^2 IR_{\lambda,l}^{m,n+1} f(z)}{(IR_{\lambda,l}^{m,n} f(z))^{1+\delta}} < \frac{1 + A_2 z}{1 + B_2 z}, \quad z \in U,
\]
hence $\frac{1 + A_1 z}{1 + B_1 z}$ and $\frac{1 + A_2 z}{1 + B_2 z}$ are the best subordinant and the best dominant, respectively.

For $q_1(z) = \left(\frac{1 + z}{1 - z}\right)^{\gamma_1}$, $q_2(z) = \left(\frac{1 + z}{1 - z}\right)^{\gamma_2}$, where $0 < \gamma_1 < \gamma_2 \leq 1$, we have the following corollary.
Competing interests
The author declares that she has no competing interests.

Author’s contributions
The author drafted the manuscript, read and approved the final manuscript.

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