Some strong and $\triangle$-convergence theorems for multi-valued mappings in hyperbolic spaces

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Abstract: We introduce an iteration process for three multi-valued mappings in hyperbolic spaces and establish the strong and $\triangle$-convergence theorems using the new iteration process. The results presented in this paper extend, unify and generalize some previous works from the current existing literature.

Keywords: Hyperbolic space, multi-valued mappings, common fixed point, $\triangle$-convergence, strong convergence.

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1 Introduction

Let $K$ be a nonempty subset of a metric space $(X,d)$. The set $K$ is called proximinal if for any $x \in X$, there exists an element $k \in K$ such that $d(x,k) = d(x,K)$, where $d(x,K) = \inf \{d(x,y) : y \in K\}$. We shall denote $CB(K)$ and $P(K)$ be the family of nonempty closed bounded all subsets and nonempty proximinal bounded all subsets of $K$, respectively. The Hausdorff metric on $CB(X)$ is defined by

$$H(A,B) = \max \left\{ \sup_{x \in A} d(x,B), \sup_{y \in B} d(y,A) \right\} \text{ for all } A,B \in CB(X).$$

Let $T : K \to CB(K)$ be a multi-valued mapping. An element $p \in K$ is a fixed point of $T$ if $p \in Tp$. Denote by $F(T)$ the set of all fixed points of $T$ and $P_T(x) = \{y \in Tx : d(x,y) = d(x,Tx)\}$. It follows from the definition of $P_T$ that $d(x,Tx) \leq d(x,P_T(x))$ for any $x \in K$. The mapping $T$ is said to be

(i) nonexpansive if $H(Tx,Ty) \leq d(x,y)$ for all $x,y \in K$;
(ii) quasi-nonexpansive [17] if $F(T) \neq \emptyset$ and $H(Tx,Tp) \leq d(x,p)$ for all $x \in K$ and $p \in F(T)$;
(iii) Lipschitzian if there exists a constant $L > 0$ such that $H(Tx,Ty) \leq Ld(x,y)$ for all $x,y \in K$;
(iv) Lipschitzian quasi-nonexpansive if both (ii) and (iii) hold.

It is clear that each multi-valued nonexpansive mapping with $F(T) \neq \emptyset$ is quasi-nonexpansive. But there exist the multi-valued quasi-nonexpansive mappings that are not nonexpansive (see [16, 17]). Moreover, each multi-valued nonexpansive mapping is Lipschitzian with $L = 1$.

Agarwal, O’Regan and Sahu [1] introduced the following iteration process, which is independent of both Mann [13] and Ishikawa [7] iterations, for a single-valued nonexpansive mapping in a Banach space:

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n f(x_n) \quad \text{for all } n \geq 0$$

where $f : X \to X$ is a single-valued nonexpansive mapping and $\{\alpha_n\} \subseteq (0,1)$ is a sequence.

In this paper, we introduce an iteration process for three multi-valued mappings in hyperbolic spaces and establish the strong and $\triangle$-convergence theorems using the new iteration process. The results presented in this paper extend, unify and generalize some previous works from the current existing literature.
A hyperbolic space \([10]\) is a triple \((X,d,W)\) where \(X,d,W\) are the concept of hyperbolic space in Reich and Shafrir \([14]\). Our results generalize some recent results given in \([8,15]\).

In this paper, we prove some convergence theorems of the iteration process (2) for approximating a common fixed point of three multi-valued Lipschitzian quasi-nonexpansive mappings in a hyperbolic space. The class of hyperbolic space also contains Hadamard manifolds (see \([3]\)), the CAT(0) space in the sense of Gromov (see \([2]\)) and Banach space are the examples of hyperbolic space. The concept of hyperbolic space is more restrictive than the hyperbolic type introduced in Goebel and Kirk \([4]\) and more general than the hyperbolic space introduced by Kohlenbach \([10]\).\]  

2 Preliminaries and lemmas

We consider the concept of hyperbolic space introduced by Kohlenbach \([10]\) which is more restrictive than the hyperbolic type introduced in Goebel and Kirk \([4]\) and more general than the concept of hyperbolic space in Reich and Shafrir \([14]\).

A hyperbolic space \([10]\) is a triple \((X,d,W)\) where \((X,d)\) is a metric space and \(W : X \times X \times [0,1] \rightarrow X\) is a function satisfying

(W1) \(d(z,W(x,y,\lambda)) \leq (1-\lambda)d(z,x) + \lambda d(z,y)\),
(W2) \(d(W(x,y,\lambda_1),W(x,y,\lambda_2)) = |\lambda_1 - \lambda_2| \ d(x,y)\),
(W3) \(W(x,y,\lambda) = W(y,x,(1-\lambda))\),
(W4) \(d(W(x,z,\lambda),W(y,w,\lambda)) \leq (1-\lambda)d(x,y) + \lambda d(z,w)\)

for all \(x,y,z,w \in X\) and \(\lambda,\lambda_1,\lambda_2 \in [0,1]\).

If a space satisfies only (W1), it coincides with the convex metric space introduced by Takahashi \([20]\). A subset \(K\) of a hyperbolic space \(X\) is convex if \(W(x,y,\lambda) \in K\) for all \(x,y \in K\) and \(\lambda \in [0,1]\). CAT(0) space in the sense of Gromov (see \([2]\)) and Banach space are the examples of hyperbolic space. The class of hyperbolic space also contains Hadamard manifolds (see \([3]\)), the Hilbert balls equipped with the hyperbolic metric (see \([5]\)), Cartesian products of Hilbert balls and \(\mathbb{R}\)-trees, as special cases.

A hyperbolic space \((X,d,W)\) is said to be uniformly convex \([18]\) if for all \(u,x,y \in X\), \(r > 0\) and \(\varepsilon \in (0,2]\), there exists a constant \(\delta \in (0,1]\) such that \(d(W(x,y,\frac{1}{2}),u) \leq (1-\delta)r\) whenever \(d(x,u) \leq r\), \(d(y,u) \leq r\) and \(d(x,y) \geq \varepsilon r\).
A mapping \( \eta : (0, \infty) \times (0, 2] \to (0, 1] \) is called the modulus of uniform convexity if \( \delta = \eta(r, \varepsilon) \) for given \( r > 0 \) and \( \varepsilon \in (0, 2] \). The function \( \eta \) is monotone if it decreases with \( r \) (for a fixed \( \varepsilon \)).

Let \( \{x_n\} \) be a bounded sequence in a metric space \( X \). For \( x \in X \), define a continuous functional \( r(., \{x_n\}) : X \to [0, \infty) \) by

\[
r(x, \{x_n\}) = \limsup_{n \to \infty} d(x, x_n).
\]

The asymptotic radius \( r_K(\{x_n\}) \) of \( \{x_n\} \) with respect to a subset \( K \) of \( X \) is given by

\[
r_K(\{x_n\}) = \inf \{ r(x, \{x_n\}) : x \in K \}.
\]

The asymptotic center \( A_K(\{x_n\}) \) of \( \{x_n\} \) with respect to \( K \subset X \) is the set

\[
A_K(\{x_n\}) = \{ x \in K : r(x, \{x_n\}) = r_K(\{x_n\}) \}.
\]

\( r(\{x_n\}) \) and \( A(\{x_n\}) \) will denote the asymptotic radius and the asymptotic center of \( \{x_n\} \) with respect to \( X \), respectively. In general, the set \( A_K(\{x_n\}) \) may be empty or may even contain infinitely many points. It has been shown in Proposition 3.3 of [11] that every bounded sequences have unique asymptotic center with respect to nonempty closed convex subsets in a complete uniformly convex hyperbolic space with the monotone modulus of uniform convexity.

A sequence \( \{x_n\} \) in \( X \) is said to be \( \triangle \)-convergent to \( x \in X \) if \( x \) is the unique asymptotic center of \( \{u_n\} \) for every subsequence \( \{u_n\} \) of \( \{x_n\} \) (see [12]). In this case, we write \( \triangle \)-lim_{n \to \infty} x_n = x \) and call \( x \) as \( \triangle \)-limit of \( \{x_n\} \).

In the sequel, we shall need the following results.

**Lemma 1** (see [9, Lemma 2.5]) Let \( (X, d, W) \) be a uniformly convex hyperbolic space with the monotone modulus of uniform convexity \( \eta \). Let \( x \in X \) and \( \{\alpha_n\} \) be a sequence in \([a, b]\) for some \( a, b \in (0, 1) \). If \( \{x_n\} \) and \( \{y_n\} \) are sequences in \( X \) such that \( \limsup_{n \to \infty} d(x_n, x) \leq r, \limsup_{n \to \infty} d(y_n, x) \leq r \) and \( \lim_{n \to \infty} d(W(x_n, y_n, \alpha_n), x) = r \) for some \( r \geq 0 \), then

\[
\lim_{n \to \infty} d(x_n, y_n) = 0.
\]

**Lemma 2** (see [9, Lemma 2.6]) Let \( K \) be a nonempty closed convex subset of a uniformly convex hyperbolic space \( X \) and \( \{x_n\} \) be a bounded sequence in \( K \) with \( A(\{x_n\}) = \{y\} \). If \( \{y_m\} \) is another sequence in \( K \) such that \( \lim_{m \to \infty} r(y_m, \{x_n\}) = r(y, \{x_n\}) \), then \( \lim_{m \to \infty} y_m = y. \)

**Lemma 3** (see [19, Lemma 1]) Let \( K \) be a nonempty subset of a metric space \( (X, d) \) and \( T : K \to P(K) \) be a multi-valued mapping. Then the followings are equivalent:

(1) \( x \in F(T) \), that is, \( x \in Tx \);
(2) \( P_T(x) = \{x\} \), that is, \( x = y \) for each \( y \in P_T(x) \);
(3) \( x \in F(P_T) \), that is, \( x \in P_T(x) \).

Further, \( F(T) = F(P_T) \).

### 3 Main results

From now on for three multi-valued mappings \( Q, S \) and \( T \), we set \( F = F(Q) \cap F(S) \cap F(T) \neq \emptyset \).

We start with proving key lemmas for later use.

**Lemma 4** Let \( K \) be a nonempty closed convex subset of a hyperbolic space \( X \) and \( Q, S, T : K \to P(K) \) be three multi-valued mappings such that \( P_Q, P_S \) and \( P_T \) are quasi-nonexpansive. Then for the sequence \( \{x_n\} \) defined by (2), \( \lim_{n \to \infty} d(x_n, p) \) exists for each \( p \in F \).
Proof. Let \( p \in F \). Then by Lemma 3, \( p \in P_Q(p) = \{p\} = P_S(p) = P_T(p) \). From (2), we have

\[
d(x_{n+1}, p) = d(W(u_n, v_n, \alpha_n), p) \\
\leq (1 - \alpha_n)d(u_n, p) + \alpha_n d(v_n, p) \\
(1 - \alpha_n)d(u_n, P_T(p)) + \alpha_n d(v_n, P_S(p)) \\
(1 - \alpha_n)H(P_T(y_n), P_T(p)) + \alpha_n H(P_S(x_n), P_S(p)) \\
\leq (1 - \alpha_n)d(y_n, p) + \alpha_n d(x_n, p)
\]

and

\[
d(y_n, p) = d\left(W\left(t_n, x_n, \frac{\beta_n}{1 - \alpha_n}\right), p\right) \\
\leq \left(1 - \frac{\beta_n}{1 - \alpha_n}\right)d(t_n, p) + \frac{\beta_n}{1 - \alpha_n}d(x_n, p) \\
\leq \left(1 - \frac{\beta_n}{1 - \alpha_n}\right)H(P_Q(x_n), P_Q(p)) + \frac{\beta_n}{1 - \alpha_n}d(x_n, p) \\
\leq \left(1 - \frac{\beta_n}{1 - \alpha_n}\right)d(x_n, p) + \frac{\beta_n}{1 - \alpha_n}d(x_n, p) \\
= d(x_n, p).
\]

Combining (3) and (4), we get

\[
d(x_{n+1}, p) \leq d(x_n, p).
\]

Hence \( \lim_{n \to \infty} d(x_n, p) \) exists for each \( p \in F \). □

Lemma 5. Let \( K \) be a nonempty closed convex subset of a uniformly convex hyperbolic space \( X \) with the monotone modulus of uniform convexity \( \eta \) and \( Q, S, T: K \to P(K) \) be three multi-valued mappings such that \( P_Q, P_S \) and \( P_T \) are Lipschitzian quasi-nonexpansive with \( d(x_n, v_n) \leq d(u_n, v_n) \). Let \( \{x_n\} \) be the sequence defined by (2) with \( 0 < a \leq \alpha_n, \beta_n \leq b < 1 \). Then

\[
\lim_{n \to \infty} d(x_n, P_Q(x_n)) = \lim_{n \to \infty} d(x_n, P_S(x_n)) = \lim_{n \to \infty} d(x_n, P_T(x_n)) = 0.
\]

Proof. By Lemma 4, \( \lim_{n \to \infty} d(x_n, p) \) exists for each given \( p \in F \). We assume that

\[
\lim_{n \to \infty} d(x_n, p) = r \quad \text{for some } r \geq 0.
\]

The case \( r = 0 \) is trivial. Next, we deal with the case \( r > 0 \). Now (3) can be rewritten as

\[
(1 - \alpha_n)d(x_{n+1}, p) \leq (1 - \alpha_n)d(y_n, p) + \alpha_n d(x_n, p) - \alpha_n d(x_{n+1}, p).
\]

This implies that

\[
d(x_{n+1}, p) \leq d(y_n, p) + \frac{\alpha_n}{1 - \alpha_n}[d(x_n, p) - d(x_{n+1}, p)] \\
\leq d(y_n, p) + \frac{b}{1 - b}[d(x_n, p) - d(x_{n+1}, p)]
\]

and so \( r \leq \liminf_{n \to \infty} d(y_n, p) \). Taking limit superior on both sides in the inequality (4), we get

\[
\limsup_{n \to \infty} d(y_n, p) \leq r. \quad \text{Hence}
\]

\[
\lim_{n \to \infty} d(y_n, p) = \lim_{n \to \infty} d\left(W\left(t_n, x_n, \frac{\beta_n}{1 - \alpha_n}\right), p\right) = r.
\]
Since
\[ d(t_n, p) \leq H(P_Q(x_n), P_Q(p)) \leq d(x_n, p), \]
then we have
\[ \limsup_{n \to \infty} d(t_n, p) \leq r. \]  \hfill (7)

From (5)-(7) and Lemma 1, we obtain
\[ \lim_{n \to \infty} d(t_n, x_n) = 0. \]  \hfill (8)

Since \( d(x, P_Q(x)) = \inf_{z \in P_Q(x)} d(x, z) \), therefore
\[ d(x_n, P_Q(x_n)) \leq d(x_n, t_n) \to 0 \quad \text{as } n \to \infty. \]

By (4) and the quasi-nonexpansiveness of \( P_T \), we have
\[ d(u_n, p) \leq H(P_T(y_n), P_T(p)) \leq d(y_n, p) \leq d(x_n, p). \]
Hence
\[ \limsup_{n \to \infty} d(u_n, p) \leq r. \]  \hfill (9)

Since
\[ d(v_n, p) \leq H(P_S(x_n), P_S(p)) \leq d(x_n, p), \]
then we have
\[ \limsup_{n \to \infty} d(v_n, p) \leq r. \]  \hfill (10)

In addition,
\[ \lim_{n \to \infty} d(x_{n+1}, p) = \lim_{n \to \infty} d(W(u_n, v_n, \alpha_n), p) = r. \]  \hfill (11)

From (9)-(11) and Lemma 1, we obtain
\[ \lim_{n \to \infty} d(u_n, v_n) = 0. \]

Hence, from the hypothesis \( d(x_n, v_n) \leq d(u_n, v_n) \), we have
\[ d(x_n, P_S(x_n)) \leq d(x_n, v_n) \leq d(u_n, v_n) \to 0 \quad \text{as } n \to \infty. \]

Since
\[ d(x_n, u_n) \leq d(x_n, v_n) + d(v_n, u_n) \leq 2d(u_n, v_n) \to 0 \quad \text{as } n \to \infty, \]  \hfill (12)
we conclude that
\[ d(x_n, P_T(y_n)) \leq d(x_n, u_n) \to \infty \quad \text{as } n \to \infty. \]

In addition, by (8) and (12), we get
\[
\begin{align*}
\frac{d(x_n, P_T(x_n))}{d(x_n, u_n) + d(u_n, P_T(x_n))} & \leq d(x_n, u_n) + H(P_T(y_n), P_T(x_n)) \\
& \leq d(x_n, u_n) + Ld(y_n, x_n) \\
& \leq d(x_n, u_n) + L \left(1 - \frac{\beta_n}{1 - \alpha_n}\right) d(t_n, x_n) \\
& \leq d(x_n, u_n) + L \left(1 - \frac{\alpha_n}{1 - a}\right) d(t_n, x_n) \\
& \to 0 \quad \text{as } n \to \infty.
\end{align*}
\]

This completes the proof. \( \square \)

We now give our \( \triangle \)-convergence theorem.
**Theorem 6** Let $X, K$ and $\{x_n\}$ satisfy the hypotheses of Lemma 5 and $Q, S, T : K \to P(K)$ be three multi-valued mappings such that $P_Q, P_S$ and $P_T$ are nonexpansive. If $X$ is complete, then the sequence $\{x_n\}$ is $\Delta$-convergent to a point in $F$.

**Proof.** It follows from Lemma 4 that the sequence $\{x_n\}$ is bounded. Then $\{x_n\}$ has a unique asymptotic center $A_K (\{x_n\}) = \{x\}$. Let $\{z_n\}$ be any subsequence of $\{x_n\}$ with $A_K (\{z_n\}) = \{z\}$. By Lemma 5, we have
\[
\lim_{n \to \infty} d(z_n, P_Q(z_n)) = \lim_{n \to \infty} d(z_n, P_S(z_n)) = \lim_{n \to \infty} d(z_n, P_T(z_n)) = 0.
\]

Now, we claim that $z$ is a common fixed point of $P_Q, P_S$ and $P_T$. For this, we define a sequence $\{w_m\}$ in $P_T(z)$. So, we calculate
\[
d(w_m, z_n) \leq d(w_m, P_T(z_n)) + d(P_T(z_n), z_n)
\]
\[
\leq H(P_T(z), P_T(z_n)) + d(P_T(z_n), z_n)
\]
\[
\leq d(z, z_n) + d(P_T(z_n), z_n).
\]

Then
\[
r(w_m, \{z_n\}) = \limsup_{n \to \infty} d(w_m, z_n) \leq \limsup_{n \to \infty} d(z, z_n) = r(z, \{z_n\}).
\]

This implies that $|r(w_m, \{z_n\}) - r(z, \{z_n\})| \to 0$ as $m \to \infty$. It follows from Lemma 2 that $\lim_{m \to \infty} w_m = z$. Note that $Tz \in P(K)$ being proximinal is closed, hence $P_T(z)$ is closed. Consequently $\lim_{m \to \infty} w_m = z \in P_T(z)$ and so $z \in F(P_T)$. Similarly, $z \in F(P_S)$ and $z \in F(P_Q)$. Hence $z \in F$. By the uniqueness of asymptotic center, we can get $x = z$. It implies that the sequence $\{x_n\}$ is $\Delta$-convergent to $x \in F$. The proof is completed. \(\blacksquare\)

**Remark 1** If we take $Q = S$ in Theorem 6, we get the $\Delta$-convergence theorem in [8].

**Theorem 7** Let $X, K, Q, S, T \{x_n\}$ be the same as in Lemma 5. Then
\[(i) \liminf_{n \to \infty} d(x_n, F) = \limsup_{n \to \infty} d(x_n, F) = 0 \text{ if } \{x_n\} \text{ converges strongly to a common fixed point in } F.
\]
\[(ii) \{x_n\} \text{ converges strongly to a common fixed point in } F \text{ if } X \text{ is complete and either } \liminf_{n \to \infty} d(x_n, F) = 0 \text{ or } \limsup_{n \to \infty} d(x_n, F) = 0.
\]

**Proof.** (i) Let $p \in F$. Since $\{x_n\}$ converges strongly to $p$, $\lim_{n \to \infty} d(x_n, p) = 0$. So, for a given $\epsilon > 0$, there exists $n_0 \in N$ such that $d(x_n, p) < \epsilon$ for all $n \geq n_0$. Taking infimum over $p \in F$, we get
\[
d(x_n, F) < \epsilon \quad \text{for all } n \geq n_0.
\]

This means $\lim_{n \to \infty} d(x_n, F) = 0$ so that
\[
\liminf_{n \to \infty} d(x_n, F) = \limsup_{n \to \infty} d(x_n, F) = 0.
\]

(ii) Suppose that $X$ is complete and $\liminf_{n \to \infty} d(x_n, F) = 0$ or $\limsup_{n \to \infty} d(x_n, F) = 0$. It follows from Lemma 4 that $\lim_{n \to \infty} d(x_n, F)$ exists. Then, we get
\[
\lim_{n \to \infty} d(x_n, F) = 0.
\]

The proof of the remaining part follows the proof of Theorem 2.5 in [8]. \(\blacksquare\)

Recall that a multi-valued mapping $T : K \to P(K)$ is *semi-compact* if any bounded sequence $\{x_n\}$ satisfying $d(x_n, Tx_n) \to 0$ as $n \to \infty$ has a strongly convergent subsequence.
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Gu and He [6] defined the concept of condition \((A')\) for \(N\) multi-valued mappings. We can define this concept for three multi-valued mappings as follows.

The mappings \(Q, S\) and \(T\) are said to satisfy condition \((A')\) if there exists a non-decreasing function \(f : [0, \infty) \rightarrow [0, \infty)\) with \(f(0) = 0\) and \(f(r) > 0\) for all \(r \in (0, \infty)\) such that

\[
f(d(x, F)) \leq \frac{1}{3} [d(x, Qx) + d(x, Sx) + d(x, Tx)] \quad \text{for all} \quad x \in K.
\]

By using the above definitions, we can easily prove the following strong convergence result.

**Theorem 8** Let \(X, K, Q, S, T\) and \(\{x_n\}\) be satisfy the hypotheses of Lemma 5 and \(X\) be a complete. If one of the mappings \(P_Q, P_S\) and \(P_T\) is semi-compact or \(P_Q, P_S\) and \(P_T\) satisfy condition \((A')\), then the sequence \(\{x_n\}\) is convergent strongly to a point in \(F\).

**Remark 2** (i) Theorems 7, 8 contain the corresponding results of Khan and Abbas [8] when \(S, T\) are two multi-valued mappings such that \(P_S\) and \(P_T\) are nonexpansive and \(Q = S\).

(ii) Our results generalize the corresponding results of Şahin and Başarır [15] from three nonexpansive self mappings to three multi-valued Lipschitzian quasi-nonexpansive mappings.

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**References**


