Generalization on local property of absolute matrix summability of factored Fourier series

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Abstract: In this paper, a known theorem dealing with $|\hat{N}, p_n|_k$ summability methods of Fourier series is generalized to more general cases by taking normal matrices and by using local property of absolute matrix summability of factored Fourier series.

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1 Introduction

Let $(s_n)$ denote the $n$-th partial sum of the series $\sum a_n$. We write

$$R_n = \left\{ s_1 + \frac{1}{2}s_2 + \frac{1}{3}s_3 + \ldots + \frac{1}{n}s_n \right\} / \log n.$$

Then the series $\sum a_n$ is said to be absolutely summable $(R, \log n, 1)$ or summable $|R, \log n, 1|$ if the sequence $\{R_n\}$ is of bounded variation, that is, the infinite series

$$\sum |R_n - R_{n+1}|$$

is convergent. Let $(p_n)$ be a sequence of positive numbers such that

$$P_n = \sum_{v=0}^{n} p_v \to \infty \ \text{as} \ n \to \infty, \ \ (P_{-i} = p_{-i} = 0, \ i \geq 1).$$

The sequence-to-sequence transformation

$$w_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v s_v$$

defines the sequence $(w_n)$ of the Riesz mean or simply the $(\hat{N}, p_n)$ mean of the sequence $(s_n)$ generated by the sequence of coefficients $(p_n)$ (see [8]).

The series $\sum a_n$ is said to be summable $|\hat{N}, p_n|_k$, $k \geq 1$, if (see [3])

$$\sum_{n=1}^{\infty} \left( \frac{P_n}{p_n} \right)^{k-1} |w_n - w_{n-1}|^k < \infty.$$
In the special case when \( p_n = 1 \) for all values of \( n \) (resp.\( k = 1 \)), \(|N,p_n|_k\) summability is the same as \(|C,1|_k\) (resp.\(|N,p_n|)\) summability. Also, if we take \( k = 1 \) and \( p_n = 1/(n + 1) \), \(|N,p_n|_k\) summability is equivalent to \(|R,\log n,1|\) summability.

A lower triangular matrix of nonzero diagonal entries is said to be a normal matrix. Let \( A = (a_{nv}) \) be a normal matrix, we associate two lower semimatrices \( \tilde{A} = (\tilde{a}_{nv}) \) and \( \hat{A} = (\hat{a}_{nv}) \) with entries defined by,

\[
\tilde{a}_{nv} = \sum_{i=v}^{n} a_{ni}, \quad n, v = 0, 1, \ldots
\]

and

\[
\hat{a}_{00} = a_{00}, \quad \hat{a}_{nv} = \Delta \tilde{a}_{nv}, \quad n = 1, 2, \ldots
\]

It should be noted that \( \hat{A} \) and \( \tilde{A} \) are the well-known matrices of series to series and series to sequence transformations, respectively. Then, we have

\[
A_n(s) = \sum_{v=0}^{n} a_{nv}s_v = \sum_{v=0}^{n} \tilde{a}_{nv}a_v
\]

\[
\hat{A}_n(s) = \sum_{v=0}^{n} \hat{a}_{nv}a_v
\]

Let \( (\theta_n) \) be any sequence of positive real numbers. The series \( \sum a_n \) is said to be summable \(|A,\theta_n|, k \geq 1\), (see [12],[20]) if

\[
\sum_{n=1}^{\infty} \theta_n^{k-1} |A_n(s) - A_{n-1}(s)|^k < \infty.
\]

In the special case, if we take \( a_{nv} = \frac{p_v}{p_n} \) and \( \theta_n = \frac{p_n}{p_n} \), then we have \(|N,p_n|k\) summability. Also, if we take \( \theta_n = n \) and \( a_{nv} = \frac{p_v}{p_n} \), then we have \(|R,p_n|k\) summability (see [5]).

2 The Known Results

Let \( f \) be a periodic function with period \( 2\pi \) and integrable \((L)\) over \((-\pi, \pi)\). Without any loss of generality the constant term in the constant term in the Fourier series of \( f \) can be taken to be zero, so that

\[
f(t) \sim \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt) = \sum_{n=1}^{\infty} C_n(t).
\]

where

\[
a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t)dt, \quad a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t)\cos(nt)dt, \quad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t)\sin(nt)dt.
\]

We write

\[
\phi(t) = \frac{1}{2} \{ f(x + t) + f(x - t) \}.
\]
It is well known that the convergence of the Fourier series at \( t = x \) is a local property of \( f \) (i.e., depends only on the behaviour of \( f \) in an arbitrarily small neighbourhood of \( x \)), and so the summability of the Fourier series \( t = x \) by any regular linear summability method is also a local property of \( f \).

It has been pointed out by Bosanquet [1] that for the case \( \lambda_n = \log n \), the definition of absolutely summable \( (R, \log n, 1) \) or summable \( [R, \log n, 1] \) is equivalent to the definition of the summability \( [R, \lambda_n, 1] \) used by Mohanty [11], \( \lambda_n \) being a monotonic increasing sequence tending to infinity with \( n \).

Matsumoto [9] improved this result by replacing the series \( \sum (\log n)^{-1} C_n(t) \) by

\[
\sum (\log \log n)^{-p} C_n(t), \quad p > 1.
\]

Bhatt [2] showed that the factor \((\log \log n)^{-p}\) in the above series can be replaced by the more general factor \(\gamma_n \log n\) where \((\gamma_n)\) is a convex sequence such that \(\sum n^{-1} \gamma_n\) is convergent. Borwein [7] generalized Bhatt’s result by proving that \((\lambda_n)\) is a sequence for which

\[
\sum_{n=1}^{\infty} \frac{p_n}{P_n} |\lambda_n| < \infty \quad \text{and} \quad \sum_{n=1}^{\infty} |\Delta \lambda_n| < \infty,
\]

then the summability \([R, P_n, 1]\) of the factored Fourier series

\[
\sum_{n=1}^{\infty} \lambda_n C_n(t)
\]

at any point is a local property of \( f \). On the other hand, Mishra [10] proved that if \((\gamma_n)\) is as above, and if

\[
P_n = O(np_n) \quad \text{and} \quad P_n \Delta p_n = O(p_n p_{n+1}),
\]

the summability \([\bar{N}, p_n]\) of the series

\[
\sum_{n=1}^{\infty} \gamma_n \frac{P_n}{np_n} C_n(t),
\]

at any point is a local property of \( f \). Bor [4] showed that \([\bar{N}, p_n]\) in Mishra’s result can be replaced by a more general summability method \([\bar{N}, p_n]_k\), and introduced the following theorem on the local property of the summability \([\bar{N}, p_n]_k\) of the factored Fourier series, which generalizes most of the above results under more appropriate conditions then those given in them.

**Theorem 2.1** [6] Let \( k \geq 1 \) and the sequences \((p_n)\) and \((\lambda_n)\) be such that

1. \( \Delta X_n = O(1/n) \),
2. \( \sum_{n=1}^{\infty} n^{-1} \left( |\lambda_n|^k + |\lambda_{n+1}|^k \right) X_n^{k-1} < \infty \),
3. \( \sum_{n=1}^{\infty} (X_n^k + 1) |\Delta \lambda_n| < \infty \),

where \( X_n = (np_n)^{-1} P_n \). Then the summability \([\bar{N}, p_n]_k \) \( k \geq 1 \) of the series \( \sum_{n=1}^{\infty} \lambda_n X_n C_n(t) \) at a point can be ensured by a local property.
3 The Main Results

Many studies have been done for matrix generalization of Fourier series (see [13]-[28]). The aim of this paper is to extend Theorem 2.1 for $|A, \theta_n|_k$ summability method by taking normal matrices instead of weighted mean matrices.

**Theorem 3.1** Let $A = (a_{nv})$ be a positive normal matrix such that

\[
\overline{a}_{n0} = 1, \quad n = 0, 1, ..., \quad (4)
\]

\[
a_{n-1,v} \geq a_{nv}, \quad \text{for } n \geq v + 1, \quad (5)
\]

\[
\sum_{v=1}^{n-1} a_{nv} \hat{a}_{n,v+1} = O(a_{nn}). \quad (6)
\]

Let $(\theta_n a_{nn})$ be a non increasing sequence. If $(\lambda_n)$ and $(X_n)$ are sequences satisfying the following conditions:

\[
\sum_{n=1}^{\infty} (\theta_n a_{nn})^{k-1} n^{-1} \left\{ |\lambda_n|^k + |\lambda_{n+1}|^k \right\} X_n^{k-1} < \infty, \quad (7)
\]

\[
\sum_{n=1}^{\infty} (\theta_n a_{nn})^{k-1} (X_n^k + 1)|\Delta \lambda_n| < \infty, \quad (8)
\]

\[
\Delta X_n = O(1/n), \quad (9)
\]

where $X_n = (na_{nn})^{-1}$, and $(\theta_n)$ is any sequence of positive constants, then the summability $|A, \theta_n|_k, k \geq 1$ of the series

\[
\sum \lambda_n X_n C_n(t),
\]

at a point can be ensured by a local property.

We need the following lemma for the proof of Theorem 3.1.

**Lemma 3.2** Let $(\theta_n a_{nn})$ be a non increasing sequence. Suppose that the matrix $A$ and the sequences $(\lambda_n)$ and $(X_n)$ satisfy all the conditions of Theorem 3.1, and that $(s_n)$ is bounded and $(\theta_n)$ is any sequence of positive constants. Then the series

\[
\sum_{n=1}^{\infty} \lambda_n X_n a_n \quad (10)
\]

is summable $|A, \theta_n|_k, k \geq 1$.

4 Proof of Lemma 3.2

Let $(T_n)$ denotes the A-transform of the series (10). Then we have,

\[
\Delta T_n = \sum_{v=1}^{n} \hat{a}_{nv} \lambda_v X_v, \quad X_0 = 0.
\]

Applying Abel’s transformation to this sum we have

\[
\overline{\Delta} T_n = \sum_{v=1}^{n-1} \Delta(\hat{a}_{nv} \lambda_v X_v) s_v + a_{nn} \lambda_n X_n s_n.
\]
By the formula for the difference of products of sequences (see [8], p.129) we have
\[
\Delta(\hat{a}_{nv} \lambda_v X_v) = \lambda_v X_v \Delta \hat{a}_{nv} + \Delta(\lambda_v X_v) \hat{a}_{nv} + \Delta(\lambda_v) \hat{a}_{nv} + (X_v \Delta \lambda_v + \Delta X_v \lambda_v) \hat{a}_{nv},
\]
\[
\Delta T_n = \sum_{v=1}^{n-1} \hat{a}_{n,v+1} X_v \Delta \lambda_v s_v + \sum_{v=1}^{n-1} \hat{a}_{n,v+1} \lambda_v \Delta X_v s_v + \sum_{v=1}^{n-1} \Delta \hat{a}_{nv} \lambda_v X_v s_v + a_{nn} \lambda_v X_v s_n
\]
\[
= T_n(1) + T_n(2) + T_n(3) + T_n(4).
\]

To complete the proof of Lemma 3.2, by Minkowski inequality, it is sufficient to show that
\[
\sum_{n=1}^{\infty} \theta_n^{k-1} |T_{n,r}|^k < \infty, \quad \text{for } r = 1, 2, 3, 4. \quad (11)
\]

The elements \( \hat{a}_{nv} \geq 0 \) for each \( v, n \). It is easily seen by using conditions (4) and (5) of Theorem 3.1. For detail (see [18]).

Also,
\[
\sum_{v=1}^{n-1} |\Delta a_{nv}| = \sum_{v=1}^{n-1} (a_{n-1,v} - a_{nv}) = a_{n-1,0} - a_{n0} + a_{n0} - a_{n-1,0} + a_{nn}
\]
\[
= a_{n0} - a_{n-1,0} + a_{nn} \leq a_{nn}. \quad (12)
\]

First, by applying Hölder’s inequality with indices \( k \) and \( k' \), where \( k > 1 \) and \( \frac{1}{k} + \frac{1}{k'} = 1 \), we have that
\[
\sum_{n=2}^{m+1} \theta_n^{k-1} |T_{n,1}|^k \leq \sum_{n=2}^{m+1} \theta_n^{k-1} \left( \sum_{v=1}^{n-1} |\hat{a}_{n,v+1} X_v | \Delta \lambda_v |s_v| \right)^k
\]
\[
= O(1) \sum_{n=2}^{m+1} \theta_n^{k-1} \left( \sum_{v=1}^{n-1} |\hat{a}_{n,v+1} X_v | \Delta \lambda_v |s_v| \right)^k
\]
and by taking account of (4) and (5), we have \( \hat{a}_{n,v+1} \leq a_{nn} \), for \( 1 \leq v \leq n-1 \) which implies that
\[
\sum_{v=1}^{n-1} \hat{a}_{n,v+1} |\Delta \lambda_v| \leq a_{nn} \sum_{v=1}^{n-1} |\Delta \lambda_v| = O(a_{nn}),
\]

thus,
\[
\sum_{n=2}^{m+1} \theta_n^{k-1} |T_{n,1}|^k = O(1) \sum_{n=2}^{m+1} \theta_n^{k-1} a_{nn}^{k-1} \sum_{v=1}^{n-1} \hat{a}_{n,v+1} X_v | \Delta \lambda_v |
\]
\[
= O(1) \sum_{v=1}^{m} X_v | \Delta \lambda_v | \sum_{n=v+1}^{m+1} (\theta_v a_{nn})^{k-1} \hat{a}_{n,v+1} = O(1) \sum_{v=1}^{m} (\theta_v a_{nn})^{k-1} X_v | \Delta \lambda_v | \sum_{n=v+1}^{m+1} \hat{a}_{n,v+1}
\]
\[
= O(1) \sum_{v=1}^{m} (\theta_v a_{nn})^{k-1} X_v | \Delta \lambda_v |
\]
\[
= O(1) \quad \text{as } m \to \infty,
\]
in view of condition (8). Note that from (9) follows that \( \Delta X_v = O(a_{vv}X_v) \). Also, we have

\[
\sum_{n=2}^{m+1} \vartheta_n^{k-1} |T_{n,2}|^k \leq \sum_{n=2}^{m+1} \vartheta_n^{k-1} \left( \sum_{\nu=1}^{n-1} \hat{a}_n,\nu+1 |\nu+1||\Delta X_v||s_\nu| \right)^k
\]

\[
= O(1) \sum_{n=2}^{m+1} \vartheta_n^{k-1} \left( \sum_{\nu=1}^{n-1} \hat{a}_n,\nu+1 |\nu+1|a_{vv}X_v \right)^k
\]

\[
= O(1) \sum_{n=2}^{m+1} \vartheta_n^{k-1} \left( \sum_{\nu=1}^{n-1} \hat{a}_n,\nu+1 |\nu+1|^k a_{vv}X_v^k \right) \left( \sum_{\nu=1}^{n-1} a_{vv}\hat{a}_n,\nu+1 \right)^{k-1}
\]

\[
= O(1) \sum_{n=2}^{m+1} \vartheta_n^{k-1} a_{nn}^{k-1} \left( \sum_{\nu=1}^{n-1} a_{vv}\hat{a}_n,\nu+1 |\nu+1|^k X_v^k \right)
\]

\[
= O(1) \sum_{\nu=1}^{m} |\nu+1|^k a_{vv}X_v^k \sum_{n=\nu+1}^{m+1} (\vartheta_\nu a_{nn})^{k-1} \hat{a}_n,\nu+1 = O(1) \sum_{\nu=1}^{m} (\vartheta_\nu a_{vv})^{k-1} |\nu+1|^k a_{vv}X_v^k \sum_{n=\nu+1}^{m+1} \hat{a}_n,\nu+1
\]

\[
= O(1) \sum_{\nu=1}^{m} (\vartheta_\nu a_{vv})^{k-1} |\nu+1|^k a_{vv}X_v^k - X_v = O(1) \sum_{\nu=1}^{m} (\vartheta_\nu a_{vv})^{k-1} |\nu+1|^k - X_v^{k-1}
\]

\[
= O(1) \quad \text{as} \quad m \to \infty.
\]

by virtue of the hypotheses of Lemma 3.2. On the other hand, we have

\[
\sum_{n=2}^{m+1} \vartheta_n^{k-1} |T_{n,3}|^k = O(1) \sum_{n=2}^{m+1} \vartheta_n^{k-1} \left( \sum_{\nu=1}^{n-1} |\Delta a_{nv}||\nu+1|X_v \right)^k
\]

\[
= O(1) \sum_{n=2}^{m+1} \vartheta_n^{k-1} \left( \sum_{\nu=1}^{n-1} |\Delta a_{nv}||\nu+1|X_v^k \right) \left( \sum_{\nu=1}^{n-1} |\Delta a_{nv}| \right)^{k-1}
\]

\[
= O(1) \sum_{n=2}^{m+1} \vartheta_n^{k-1} a_{nn}^{k-1} \sum_{\nu=1}^{n-1} |\Delta a_{nv}||\nu+1|^k X_v^k
\]

\[
= O(1) \sum_{\nu=1}^{m} |\nu+1|^k X_v^k \sum_{n=\nu+1}^{m+1} (\vartheta_\nu a_{nn})^{k-1} |\Delta a_{nv}| = O(1) \sum_{\nu=1}^{m} (\vartheta_\nu a_{vv})^{k-1} |\nu+1|^k X_v^k \sum_{n=\nu+1}^{m+1} |\Delta a_{nv}|
\]

\[
= O(1) \sum_{\nu=1}^{m} (\vartheta_\nu a_{vv})^{k-1} |\nu+1|^k X_v^{k-1}a_{vv}
\]

\[
= O(1) \sum_{\nu=1}^{m} (\vartheta_\nu a_{vv})^{k-1} |\nu+1|^k X_v^{k-1} - 1
\]

\[
= O(1) \quad \text{as} \quad m \to \infty.
\]

by virtue of the hypotheses of Lemma 3.2. Finally, we have that

\[
\sum_{n=1}^{\infty} \vartheta_n^{k-1} |T_{n,4}|^k = O(1) \sum_{n=1}^{\infty} \vartheta_n^{k-1} |\nu+1|^k X_v^k a_{nn}
\]

\[
= O(1) \sum_{n=1}^{\infty} (\vartheta_\nu a_{nn})^{k-1} |\nu+1|^k X_v^k a_{nn}
\]

\[
= O(1) \sum_{n=1}^{\infty} (\vartheta_\nu a_{nn})^{k-1} |\nu+1|^k X_v^{k-1} \nu^{n-1} < \infty,
\]

by virtue of the hypotheses of Lemma 3.2. This completes the proof of Lemma 3.2.

**Proof of Theorem 3.1.** Since the convergence of the Fourier series at a point is a local property of its generating function \( f \), the theorem follows by formula (7.1) from Chapter II of the book (see [29]) and from Lemma 3.2.
5 APPLICATIONS

We can apply Theorem 3.1 to weighted mean \( A = (a_{nv}) \) is defined as 
\[
a_{nv} = \frac{p_v}{P_n}
\]
where \( P_n = p_0 + p_1 + \ldots + p_n \). We have that,
\[
\bar{a}_{nv} = \frac{P_n - P_{v-1}}{P_n} \quad \text{and} \quad \hat{a}_{n,v+1} = \frac{p_n P_v}{P_n P_{n-1}}.
\]
The following results can be easily verified.
1. If we take \( \theta_n = \frac{P_n}{p_n} \) in Theorem 3.1, then we have another theorem dealing with absolute matrix summability (see [18]).
2. If we take \( \theta_n = \frac{P_n}{p_n} \) and \( a_{nv} = \frac{p_v}{P_n} \) in Theorem 3.1, then we have a theorem dealing with \([N, p_n]_k\) summability (see [6]).
3. If we take \( \theta_n = n \) and \( a_{nv} = \frac{p_v}{P_n} \) in Theorem 3.1, then we obtain a new result dealing with \([R, p_n]_k\) summability method.
4. If we take \( \theta_n = n, a_{nv} = \frac{p_v}{P_n} \) and \( p_n = 1 \) for all values of \( n \) in Theorem 3.1, then we have a result for \([C, 1]_k\) summability.

References


