Coefficient Estimates for Initial Taylor-Maclaurin Coefficients for a Subclass of Analytic and Bi-univalent Functions Associated with $q$-Derivative Operator

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Abstract: In the present paper, we introduce and investigate a new subclass of analytic and bi-univalent functions $\Sigma_q(\varphi)$ in the open unit disk with respect to $q$-derivative operator. For functions belonging to this class, we obtain estimates on the first two Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$. Various other results, which presented in this paper, would generalize and improve those in related works of several earlier authors.

Keywords: Analytic functions, Bi-univalent functions, $q$-derivative, Coefficient estimates.

MSC2010: 30C45, 30C50.

1 Introduction

Let $\mathcal{A}$ be the class of all analytic functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k$$

in the open unit disk $\mathbb{U} = \{ z \in \mathbb{C} : |z| < 1 \}$.

An analytic function $f$ is subordinate to an analytic function $g$, written as $f \prec g$, provided there is an analytic (Schwarz) function $w$ with $w(0) = 0$, $|w(z)| < 1$, for all $z \in \mathbb{U}$ satisfying $f(z) = g(w(z))$ for all $z \in \mathbb{U}$.

The well-known Koebe one-quarter theorem [1] ensure that the image of $\mathbb{U}$ under every univalent function $f \in \mathcal{A}$ contains a disk of radius $\frac{1}{4}$. Hence, every univalent function $f$ has an inverse $f^{-1}$ satisfying $f^{-1}(f(z)) = z$, $(z \in \mathbb{U})$ and

$$f^{-1}(f(w)) = w, \quad (|w| < r_0(f), r_0(f) \geq \frac{1}{4})$$

where

$$g(w) = f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \cdots$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in $\mathbb{U}$ if both $f$ and $f^{-1}$ are univalent in $\mathbb{U}$. Let $\Sigma$ denote the class of bi-univalent functions in $\mathbb{U}$ given by (1).
In 1986, Brannan and Taha [2] introduced certain subclasses of the bi-univalent function class $\Sigma$ similar to the familiar subclasses of starlike and convex functions of order $\alpha$. In 2012, Ali et al. [3] widen the result of Brannan and Taha using subordination. Since then, various subclasses of the bi-univalent function class $\Sigma$ were introduced and non-sharp estimates on the first two coefficients $a_2$ and $a_3$ of the Taylor-Maclaurin series expansion (1) were found in several recent studies. For interesting study on this topic can be found in ([5]-[6]-[7]-[8]).

In [11], [12], Jackson defined the $q$-derivative operator $D_q$ of a function as follows:

$$D_q f(z) = \frac{f(qz) - f(z)}{(q-1)z} \quad (z \neq 0, q \neq 0)$$

(2)

and $D_q f(z) = f'(0)$. In case $f(z) = z^k$ for $k$ is a positive integer, the $q$-derivative of $f(z)$ is given by

$$D_q z^k = \frac{z^k - (zq)^k}{z(1-q)} = [k]_q z^{k-1}.$$  

As $q \to 1^-$ and $k \in \mathbb{N}$, we have

$$[k]_q = \frac{1 - q^k}{1 - q} = 1 + q + ... + q^k \to k.$$  

(3)

Quite a number of great mathematicians studied the concepts of $q$-derivative, for example by Gasper and Rahman [10], Aral et.al [13] and many others (see [15]-[20]).

Let $\varphi$ be an analytic function with positive real part in $U$ such that $\varphi(0) = 1$, $\varphi'(0) > 0$ and $\varphi(U)$ is symmetric with respect to real axis. Such a function has a series expansion of the form:

$$\varphi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots \quad (B_1 > 0).$$  

(4)

We now introduce the following subclass of analytic and bi-univalent functions using the $q$-operator.

**Definition 1.1** A function $f \in \Sigma$ is said to be in the class $\Sigma_q(\varphi)$ if each of the following subordination condition holds true:

$$D_q(f(z)) \prec \varphi(z), \quad z \in U.$$  

(5)

and

$$D_q(g(w)) \prec \varphi(w), \quad w \in U.$$  

(6)

where $g(w) = f^{-1}(w)$.

The subclass $\Sigma_q(\varphi)$ in Definition 1.1 can be reduced to many subclasses introduced before as seen in the following Remarks.

**Remark 1.2** Setting $q \to 1^-$, the class $\Sigma_q(\varphi)$ reduces to the class $\mathcal{H}_\sigma(\varphi)$ introduced by Ali et al.[3] which is a subclass of the functions $f \in \Sigma$ satisfying

$$f'(z) \prec \varphi(z), \quad g'(w) \prec \varphi(z)$$

**Remark 1.3** Setting $q \to 1^-$ and

$$\varphi(z) = \frac{1 + (1 - 2\beta)}{1 - z} \quad (0 \leq \beta < 1), \quad \varphi(z) = \left(\frac{1 + z}{1 - z}\right)^\alpha \quad (0 < \alpha \leq 1),$$

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the class \( \Sigma_q(\varphi) \) reduces to the classes \( \mathcal{H}_q^{\alpha} \) and \( \mathcal{H}_q(\beta) \) introduced by Srivastava et al.[4] which are subclasses of the functions \( f \in \Sigma \) satisfying

\[
|\arg(f'(z))| < \frac{\alpha \pi}{2}, \quad |\arg(g'(w))| < \frac{\alpha \pi}{2}
\]

and

\[
\Re(f'(z)) > \beta, \quad \Re(g'(w)) > \beta
\]

respectively.

Remark 1.4 Setting

\[
\varphi(z) = \frac{1 + (1 - 2\beta)}{1 - z} \quad (0 \leq \beta < 1) \quad \text{and} \quad \varphi(z) = \left( \frac{1 + z}{1 - z} \right)^\alpha \quad (0 < \alpha \leq 1)
\]

the class \( \Sigma_q(\varphi) \) reduces to the classes \( \mathcal{H}_q^{\alpha,\beta} \) and \( \mathcal{H}_q^{\beta} \) introduced by Bulut[9] which are subclasses of the functions \( f \in \Sigma \) satisfying

\[
|\arg(D_q f(z))| < \frac{\alpha \pi}{2}, \quad |\arg(D_q g(w))| < \frac{\alpha \pi}{2}
\]

and

\[
\Re(D_q f(z)) > \beta, \quad \Re(D_q g(w)) > \beta
\]

respectively.

In our investigation, we shall need the following Lemma

Lemma 1.5 [14] Let the function \( p \in \mathcal{P} \) be given by the following series:

\[
p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots \quad (z \in \mathbb{U}).
\]

The sharp estimate given by

\[
|p_n| \leq 2 \quad (n \in \mathbb{N}),
\]

holds true.

The object of the present paper is to find estimates on the Taylor-Maclaurin coefficients \( |a_2| \) and \( |a_3| \) for functions in this new subclass \( \Sigma_q(\varphi) \) of the function class \( \Sigma \).

2 A set of main results

For functions in the class \( \Sigma_q(\varphi) \), the following result is obtained.

Theorem 2.1 Let \( f \in \Sigma_q(\varphi) \) be of the form (1). Then

\[
|a_2| \leq \min \left\{ \frac{B_1}{[2]_q}, \frac{B_3^2}{\sqrt{[3]_q B_1^2 + [2]_q^2 (B_1 - B_2)}} \right\}
\]

and

\[
|a_3| \leq \min \left\{ \frac{B_2}{[3]_q}, \frac{B_1}{[3]_q} + \frac{B_3^2}{[2]_q} \right\}
\]

where the coefficients \( B_1 \) and \( B_2 \) are given as in (4).
Proof. Let \( f \in \Sigma_q(\varphi) \) and \( g = f^{-1} \). Then there are analytic functions \( u, v : U \to U \) with \( u(0) = v(0) = 0 \), satisfying the following conditions:

\[
D_q(f(z)) = \varphi(u(z)), \quad z \in U
\]

(9) and

\[
D_q(g(w)) = \varphi(v(w)), \quad w \in U
\]

(10)

Define the functions \( p \) and \( q \) by

\[
p(z) = \frac{1 + u(z)}{1 - u(z)} = 1 + p_1 z + p_2 z^2 + \cdots
\]

(11)

and

\[
q(z) = \frac{1 + v(z)}{1 - v(z)} = 1 + q_1 z + q_2 z^2 + \cdots
\]

(12)

Then \( p \) and \( q \) are analytic in \( U \) with \( p(0) = q(0) = 1 \). Since \( u, v : U \to U \), each of the functions \( p \) and \( q \) has a positive real part in \( U \). Therefore, in view of the above Lemma, we have

\[
\left| p_n \right| \leq 2 \quad \text{and} \quad \left| q_n \right| \leq 2 \quad (n \in \mathbb{N}).
\]

(13)

Solving for \( u(z) \) and \( v(z) \), we get

\[
u(z) = \frac{p(z) - 1}{p(z) + 1} = \frac{1}{2} \left[ p_1 z + \left( p_2 - \frac{p_1^2}{2} \right) z^2 \right] + \cdots \quad (z \in U)
\]

(14)

and

\[
v(z) = \frac{q(z) - 1}{q(z) + 1} = \frac{1}{2} \left[ q_1 z + \left( q_2 - \frac{q_1^2}{2} \right) z^2 \right] + \cdots \quad (z \in U).
\]

(15)

Upon substituting from (14) and (15) into (9) and (10), respectively, and making use of (4), we obtain

\[
D_q(f(z)) = \varphi \left( \frac{p(z) - 1}{p(z) + 1} \right) = 1 + B_1 p_1 z + \left[ \frac{1}{2} B_1 \left( p_2 - \frac{p_1^2}{2} \right) + \frac{1}{4} B_2 p_1^2 \right] z^2 + \cdots
\]

(16)

and

\[
D_q(g(w)) = \varphi \left( \frac{q(w) - 1}{q(w) + 1} \right) = 1 + B_1 q_1 w + \left[ \frac{1}{2} B_1 \left( q_2 - \frac{q_1^2}{2} \right) + \frac{1}{4} B_2 q_1^2 \right] w^2 + \cdots
\]

(17)

Equating the coefficients in (9) and (10), we find that

\[
[2]_q a_2 = \frac{1}{2} B_1 p_1
\]

(18)

\[
[3]_q a_3 = \frac{1}{2} B_1 \left( p_2 - \frac{p_1^2}{2} \right) + \frac{1}{4} B_2 p_1^2
\]

(19)

\[-[2]_q a_2 = \frac{1}{2} B_1 p_1
\]

(20)

\[
[3]_q (2a_2 - a_3) = \frac{1}{2} B_1 \left( q_2 - \frac{q_1^2}{2} \right) + \frac{1}{4} B_2 q_1^2
\]

(21)

From (18) and (20), we get

\[
p_1 = -q_1
\]

(22)
and
\[2[2]_q^2 a_2^2 = \frac{1}{4} B_1^2 (p_1^2 + q_1^2)\] (23)

Also from (19) and equation (21), we get
\[2[3]_q a_2^2 = \frac{1}{2} B_1 \left[ p_2 + q_2 - \left( \frac{p_1^2 + q_1^2}{2} \right) \right] + \frac{1}{4} B_2 |p_1^2 + q_1^2|,\] (24)

by using (23), we get
\[a_2^2 = \frac{B_1^3 (p_2 + q_2)}{4 \left( [3]_q B_1^2 + [2]_q^2 (B_1 - B_2) \right)} \] (25)

Applying Lemma 1.5 for the coefficients \(p_1, p_2, q_1, q_2\) in the equalities (23) and (25), we obtain
\[|a_2| \leq \frac{B_1^3}{\sqrt{[3]_q B_1^2 + [2]_q^2 (B_1 - B_2)}}\] (26)

Hence equations (26) and (27) gives the estimates of \(|a_2|\).

Next, in order to find the bound on \(|a_3|\), we subtract (21) from (19) and also from (22), we get \(p_1^2 = q_1^2\), hence
\[2[3]_q a_3 - 2[3]_q a_2^2 = \frac{1}{2} B_1 (p_2 - q_2),\] (28)

which, upon substitution of the value of \(a_2^2\) from (23) into (28), yields
\[a_3 = \frac{B_1}{[3]_q} (p_2 - q_2) + \frac{B_1^2}{[2]_q} (p_1^2 + q_1^2).\] (29)

So we get
\[|a_3| \leq \frac{B_1}{[3]_q} + \frac{B_1^2}{[2]_q}.\] (30)

On the other hand, upon substituting the value of \(a_2^2\) from (24) into (28), it follows that
\[a_3 = \frac{4 B_1 p_2 + (B_2 - B_1)(p_1^2 + q_1^2)}{8[3]_q}.\] (31)

And we get
\[|a_3| \leq \frac{B_2}{[3]_q}.\] (32)

Thus, we get the desired estimate on the coefficient \(|a_3|\) as asserted in (40).

### 3 Corollaries and Consequences

Taking \(q \to 1^-\) in Theorem 2.1, we obtain the following corollary. 
**Corollary 3.1** Let the function $f$ given by (1) be in the class $\Sigma(\varphi)$. Then

$$|a_2| \leq \min \left\{ \frac{B_1}{2}, \frac{B_1\sqrt{B_1}}{\sqrt{3B_1^2 + 4(B_1 - B_2)}} \right\}$$

(33)

and

$$|a_3| \leq \min \left\{ \frac{B_2}{3}, \left( \frac{1}{3} + \frac{B_1}{4} \right) B_1 \right\}$$

(34)

**Remark 3.2** Corollary 3.1 is an improvement of the following estimates obtained by Ali et al. [3].

**Corollary 3.3** (see [3]) Let the function $f$ given by (1) be in the function class $\mathcal{H}_\varphi(\varphi)$. Then

$$|a_2| \leq \frac{B_1\sqrt{B_1}}{\sqrt{3B_1^2 - 4B_2 + 4B_1}} \quad \text{and} \quad |a_3| \leq \left( \frac{1}{3} + \frac{B_1}{4} \right) B_1.$$  

(35)

Taking

$$\varphi(z) = \frac{1 + (1 - 2\beta)z}{1 - z} = 1 + 2(1 - \beta)z + 2(1 - \beta)z^2 + \cdots \quad (0 \leq \beta < 1)$$  

(36)

in Theorem 2.1, we have the following corollary.

**Corollary 3.4** [9] Let the function $f$ given by (1) be in the function class $\Sigma_q(\beta) \quad (0 \leq \beta < 1)$. Then

$$|a_2| \leq \min \left\{ \frac{2(1 - \beta)}{|2\rangle_q^2}, \sqrt{\frac{2(1 - \beta)}{|3\rangle_q^3}} \right\}$$

(37)

and

$$|a_3| \leq \frac{2(1 - \beta)}{|3\rangle_q^3}$$

(38)

Taking $q \to 1^-$ in Corollary 3.4, we have the following corollary

**Corollary 3.5** Let the function $f$ given by (1) be in the function class $\Sigma(\beta) \quad (0 \leq \beta < 1)$. Then

$$|a_2| \leq \min \left\{ (1 - \beta), \sqrt{\frac{2(1 - \beta)}{3}} \right\}$$

(39)

and

$$|a_3| \leq \frac{2(1 - \beta)}{3}$$

(40)

**Remark 3.6** Corollary 3.5 is an improvement of the following estimates obtained by Srivastave et al. [4].

**Corollary 3.7** [4] Let the function $f$ given by (1) be in the function class $\mathcal{H}_\Sigma(\alpha) \quad (0 \leq \alpha < 1)$. Then

$$|a_2| \leq \sqrt{\frac{2(1 - \alpha)}{3}}$$

(41)

and

$$|a_3| \leq \frac{(1 - \alpha)(5 - 3\alpha)}{3}.$$  

(42)
Taking
\[ \varphi(z) = \left( \frac{1 + z}{1 - z} \right)^\alpha = 1 + 2\alpha z + 2\alpha^2 z^2 + \cdots \quad (0 < \alpha \leq 1) \]
in Theorem 2.1, we have the following corollary.

**Corollary 3.8** Let the function \( f \) given by (1) be in the function class \( \Sigma_q(\alpha) \) \((0 < \alpha \leq 1)\). Then
\[ |a_2| \leq \min \left\{ \frac{2\alpha}{[2]_q}, \frac{2\alpha}{\sqrt{2[3]_q \alpha + (1 - \alpha)[2]_q^2}} \right\} \]
and
\[ |a_3| \leq \min \left\{ \frac{2\alpha^2}{[3]_q}, \frac{2\alpha}{[3]_q} + \frac{4\alpha^2}{[2]_q^2} \right\}. \]

**Remark 3.9** Corollary 3.8 is an improvement of the following estimates obtained by Bulut [9].

**Corollary 3.10** [9] Let the function \( f \) given by (1) be in the function class \( \Sigma_q(\alpha) \) \((0 < \alpha \leq 1)\). Then
\[ |a_2| \leq \frac{2\alpha}{\sqrt{2[3]_q \alpha + (1 - \alpha)[2]_q^2}} \]
and
\[ |a_3| \leq \frac{2\alpha}{[3]_q} + \frac{4\alpha^2}{[2]_q^2}. \]

Taking \( q \to 1^− \) in Corollary 3.8, we have the following corollary.

**Corollary 3.11** Let the function \( f \) given by (1) be in the function class \( \Sigma(\alpha) \) \((0 < \alpha \leq 1)\). Then
\[ |a_2| \leq \min \left\{ \alpha, \alpha \sqrt{\frac{2}{\alpha + 2}} \right\} \]
and
\[ |a_3| \leq \min \left\{ \frac{2\alpha^2}{3}, \frac{\alpha(3\alpha + 2)}{3} \right\}. \]

**Remark 3.12** Corollary 3.11 is an improvement of the following estimates obtained by Srivastave et al. [4].

**Corollary 3.13** [4] Let the function \( f \) given by (1) be in the function class \( H_{\Sigma}^q(\alpha) \) \((0 < \alpha \leq 1)\). Then
\[ |a_2| \leq \alpha \sqrt{\frac{2}{\alpha + 2}} \]
and
\[ |a_3| \leq \frac{\alpha(3\alpha + 2)}{3}. \]

**Acknowledgements.** The work here is partially supported by MOHE grant: FRGS/1/2016/STG06/UKM/01/1 and UKM grant: GUP-2017-064.
References


