ON EIGENSTATES FOR SOME SL$_2$ RELATED HAMILTONIAN

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ABSTRACT. In this paper we consider the stationary Schrödinger equation for a self-conjugated Hamiltonian $H = \frac{e + f}{i}$, where $e$ and $f$ is an anti-unitary pair of the canonical Cartan "creating" and "annihilation" operators for the classical Lie algebra $sl_2$ taken in the representation with "the lowest weight equals to 1". In this paper we prove that this operator has the continuous spectrum. Construction of eigenstates for $H$ is given in details.

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1. Introduction

This paper will deal with the representation theory of the classical Lie algebra $[1]$. We will consider the Lie algebra $sl_2$ in a certain infinitely dimensional representation corresponding to the lowest weight 1. The representation module is equivalent to the Fock Space representation of the quantum oscillator $[2]$. The "creating" and "annihilation" operators $e$ and $f$ are anti-unitary, so that the operator $H = \frac{1}{i}(e + f)$ is Hermitian, and therefore it can be interpreted as a Hamiltonian for a certain Quantum Mechanical system. This Hamiltonian is related to a Hamiltonian considered in $[3, 4]$ in the limit $q = 1$ (Note, the regime $q = 1$ was not considered in $[3, 4]$).

This paper organised as follows. In section 2 we fix the proper representation of $sl_2$ and rewrite the stationary Schrödinger equation as a linear recursion with non-constant coefficients. Section 3 is devoted to the analysis of the recursion equations. Its asymptotic is discussed in section 4. Section 5 contains discussion and conclusion.
2. Formulation of the problem

We consider the algebra $\mathfrak{sl}_2$ generated by three operators $e, f, h$ satisfying the three fundamental commutation relations [1].

$$[e, f] = h, \quad [h, e] = 2e, \quad [h, f] = -2f. \quad (1)$$

Let $\mathcal{F}$ stands for the Fock Space,

$$\mathcal{F} = \text{Span}\left\{ |n\rangle, \quad n \in \mathbb{Z}_{n \geq 0} \right\}. \quad (2)$$

The map

$$e \mapsto \pi(e) \in \text{End}(\mathcal{F}), \quad \text{etc.}, \quad (3)$$

we define as

$$e |n\rangle = |n+1\rangle i(n+1), \quad f |n\rangle = |n-1\rangle in, \quad h |n\rangle = |n\rangle (2n+1), \quad n \in \mathbb{Z}_{n \geq 0}, \quad (4)$$

where for shortness we use notation $e$ instead of $\pi(e)$, etc. Our representation (4) is the representation with the lowest weight 1,

$$h|0\rangle = |0\rangle. \quad (5)$$

(in Physics this is called ”spin $= -1/2$ representation”). The Fock co-module is defined by

$$\langle n|n'\rangle = \delta_{n, n'}, \quad n, n' \geq 0. \quad (6)$$

An essential feature of our paper is that this representation not unitary:

$$e^\dagger = -f, \quad (7)$$

where the “dagger” means the Hermitian conjugation. Subject of our interest is self-conjugated unbounded Hamiltonian

$$H = \frac{e + f}{i}, \quad (8)$$

and the stationary Schrödinger equation for it,

$$H |\psi\rangle = |\psi\rangle E. \quad (9)$$

In what follows, we will study the structure of $|\psi\rangle$ for any $E \in \mathbb{R}$ and deduce that our Hamiltonian has continuous spectrum.
3. Analysis of the recursion

We will use the Dirac notations for $\langle$bra$|$ and $|$ket$\rangle$ vectors. In components,

$$\psi_n = \langle n | \psi \rangle ,$$

(10)

where $\langle n \rangle$ is a state of Fock co-module, cf. (6), and $| \psi \rangle$ is a required wavefunction. The stationary Schrödinger equation (9) in components reads

$$(n + 1) \psi_{n+1} + n \psi_{n-1} = E \psi_n ,$$

(11)

where we assume

$$\psi_0 = 1 \quad \forall \ E \in \mathbb{R} .$$

(12)

Our aim now is to understand the asymptotic behaviour of $\psi_n$ when $n \to \infty$. Since $E$ for now is only one free parameter, we assume implicitly

$$| \psi \rangle = | \psi_E \rangle , \quad \psi_n = \psi_n(E) .$$

(13)

Recursion (11) can be identically rewritten in matrix form [3, 4]:

$$(\psi_n, \psi_{n+1}) = (\psi_{n-1}, \psi_n) \cdot L_{n+1} ,$$

(14)

where

$$L_n = \begin{pmatrix} 0 & -1 + \frac{1}{n} \\ 1 & \frac{E}{n} \end{pmatrix} .$$

(15)

Thus,

$$(\psi_{n-1}, \psi_n) = (0, 1) L_1 \cdot L_2 \cdots L_{n-1} \cdot L_n .$$

(16)

Since

$$L_\infty = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} , \quad L_4^4 = 1 ,$$

(17)

we expect mod 4 pattern for $\psi_n$. Diagonalising matrix $L_n$,

$$L_n = P_n^{-1} \begin{pmatrix} \lambda_n & 0 \\ 0 & \frac{1}{\lambda_n} \end{pmatrix} P_n ,$$

(18)

where

$$\lambda_n = i \left( \sqrt{1 - \frac{1}{n} - \frac{E^2}{4n^2} - i \frac{E}{2n}} \right) = i \sqrt{1 - \frac{1}{n}} \exp \left\{ -i \arcsin \frac{E}{2 \sqrt{n(n-1)}} \right\} ,$$

(19)
\[
P_n P_{n+1}^{-1} = 1 + \frac{1}{2n^2} \begin{pmatrix} 0 & 0 \\ -E & 1 \end{pmatrix} + O(1/n^3) , \tag{20}
\]

one can deduce the following asymptotic straightforwardly from (16):
\[
\psi_n(E) = \frac{A_n(E)}{\sqrt{n}} \cos \left( \frac{E}{2} \log n - \frac{\pi n}{2} + \varphi_n(E) \right), \quad n \gg 1 . \tag{21}
\]

Intensive numerical computations allow one to conclude that the sequences \(A_n(E)\) and \(\varphi_n(E)\) smoothly converge to \(A(E)\) and \(\varphi(E)\) when \(n \to \infty\). Therefore, we can postulate the \(1/n\) expansion for \(A_n(E)\) and \(\varphi_n(E)\):
\[
A_n(E) = A(E) \left( 1 + \frac{\delta_1}{n} + \frac{\delta_2}{n^2} + \cdots \right), \quad \varphi_n(E) = \varphi(E) + \frac{\epsilon_1}{n} + \frac{\epsilon_2}{n^2} + \cdots \tag{22}
\]

with some \(n\)-independent coefficients
\[
\delta_j = \delta_j(E), \quad \epsilon_j = \epsilon_j(E), \quad j \geq 1 . \tag{23}
\]

Values of \(\delta_j, \epsilon_j\) must follow from (11). In what follows, let us combine all correction terms in (22) into
\[
\delta(n, E) = \sum_{j=1}^{\infty} \frac{\delta_j(E)}{n^j}, \quad \epsilon(n, E) = \sum_{j=1}^{\infty} \frac{\epsilon_j(E)}{n^j} . \tag{24}
\]

To get these values, let us substitute (21) into (11). To do this in convenient way, let us introduce
\[
\Phi_n = \frac{E}{n} \log n - \frac{\pi n}{2} + \varphi_n; \quad \Phi_{n+1} = \Phi_n - \frac{\pi}{2} + \alpha_n; \quad \Phi_{n-1} = \Phi_n + \frac{\pi}{2} - \alpha' . \tag{25}
\]

The values of \(\alpha_n\) and \(\alpha'_n\) are then given by
\[
\alpha_n = \Phi_{n+1} - \Phi_n + \frac{\pi}{2} = \frac{E}{2} \log(n+1) + \varphi_{n+1} - \frac{E}{2} \log n - \varphi_n
= \frac{E}{2} \log(1 + \frac{1}{n}) + \epsilon_1\left(\frac{1}{n + 1} - \frac{1}{n}\right) + \epsilon_2\left(\frac{1}{(n+1)^2} - \frac{1}{n^2}\right) + \cdots \tag{26}
\]

and similarly for \(\alpha'_n\). Let further
\[
\frac{1}{n} = x \quad \Rightarrow \quad \frac{1}{n + 1} = \frac{x}{1 + x} = \sum_{j=1}^{\infty} (-1)^{j+1} x^j \quad \text{etc.,} \tag{27}
\]
so that 1/n-expansion becomes x-expansion. Then,

\[ \alpha_n = \frac{E}{2} \log(1 + x) + \epsilon_1 \left( \frac{x}{1 + x} - x \right) + \epsilon_2 \left( \frac{x^2}{(1 + x)^2} - x^2 \right) + \cdots \]

\[ = \frac{E}{2} x - (\frac{E}{4} + \epsilon_1)x^2 + (\frac{E}{6} + \epsilon_1 - 2\epsilon_2)x^3 + O(x^4). \] (28)

Value of \( \alpha'_n \) have similar structure.

Now we can use (25,26 and 28) in (21 and 11):

\[ \psi_n = \frac{A_n}{\sqrt{n}} \cos(\Phi_n), \]

\[ \psi_{n+1} = \frac{A_{n+1}}{\sqrt{n+1}} \cos(\Phi_n - \frac{\pi}{2} + \alpha_n) = \frac{A_{n+1}}{\sqrt{n+1}} (\sin \Phi_n \cos \alpha_n + \cos \Phi_n \sin \alpha_n) \]

\[ \psi_{n-1} = \frac{A_{n-1}}{\sqrt{n-1}} \left(- \sin \Phi_n \cos \alpha'_n + \cos \Phi_n \sin \alpha'_n \right). \] (29)

Equation (11) can be written as

\[ \cos \Phi_n \left[(n + 1) \frac{A_{n+1}}{\sqrt{n+1}} \sin \alpha_n + n \frac{A_{n-1}}{\sqrt{n-1}} \sin \alpha'_n - E \frac{A_n}{\sqrt{n}} \right] + \sin \Phi_n \left[(n + 1) \frac{A_{n+1}}{\sqrt{n+1}} \cos \alpha_n + n \frac{A_{n-1}}{\sqrt{n-1}} \cos \alpha'_n - E \frac{A_n}{\sqrt{n}} \right] = 0. \] (30)

Expressions in the square brackets are the series in 1/n. Coefficients \( \cos \Phi_n \) and \( \sin \Phi_n \) are irregular. Therefore, (30) can be satisfied if and only if:

\[(n + 1) \frac{A_{n+1}}{\sqrt{n+1}} \sin \alpha_n + n \frac{A_{n-1}}{\sqrt{n-1}} \sin \alpha'_n - E \frac{A_n}{\sqrt{n}} = 0; \]

\[(n + 1) \frac{A_{n+1}}{\sqrt{n+1}} \cos \alpha_n + n \frac{A_{n-1}}{\sqrt{n-1}} \cos \alpha'_n - E \frac{A_n}{\sqrt{n}} = 0. \] (31)

Each LHS of (31) is well defined series in \( x = 1/n \). They must be zero, so that each coefficient in \( x = 1/n \) expansion must be zero. Thus (31) provides a set of algebraic equations for \( \delta_j, \epsilon_j \).

Precise form of the asymptotic corrections is the following:

\[ \delta(n, E) = -\frac{1}{4n} + \frac{2E^2 + 1}{32n^2} - \frac{5(2E^2 - 1)}{128n^3} + \frac{20E^4 - 60E^2 - 21}{2048n^4} - \frac{180E^4 - 1380E^2 + 399}{8192n^5} + \frac{120E^6 - 2540E^4 + 2518E^2 + 869}{65536n^6} + O(n^{-7}) \] (32)
and

\[ \epsilon(n, E) = \frac{E}{4n} \left( \frac{E^2 - 5}{96n^2} + \frac{E(E^2 - 9)}{96n^3} - \frac{E(9E^4 - 490E^2 + 341)}{15360n^4} \right) \]

\[ + \frac{E(3E^4 - 190E^2 + 375)}{2560n^5} - \frac{E(15E^6 - 2793E^4 + 22169E^2 - 7615)}{258048n^6} + O(n^{-7}) \].

(33)

The correction terms \( \delta_j \) and \( \epsilon_j \) can be produced from the recursion by a bootstrap up to any order of \( 1/n \).

4. Orthogonality

There is a remarkable way to derive the inner product for two states in our model. Consider a truncated state,

\[ |\psi^{(N)}_E\rangle = \sum_{n=0}^{N} |n\rangle \psi_n(E), \]  

(34)

where \( \psi_n(E) \) are defined by (11) with the initial condition \( \psi_0 = 1 \). Straightforward computation gives

\[ \mathbf{H} |\psi^{(N)}_E\rangle = |\psi^{(N-1)}_E\rangle E + |N\rangle N\psi_{N-1}(E) + |N+1\rangle (N+1)\psi_N(E). \]  

(35)

Considering then

\[ \langle \psi^{(N)}_{E'} | \mathbf{H} |\psi^{(N)}_E\rangle, \]  

(36)

one deduces

\[ \langle \psi^{(N-1)}_{E'} | \psi^{(N-1)}_E \rangle = \frac{N}{E - E'} \left( \psi_N(E)\psi_{N-1}(E') - \psi_N(E')\psi_{N-1}(E) \right). \]  

(37)

Assuming our asymptotic for \( \psi_N \) for \( N \to \infty \), one obtains

\[ \langle \psi^{(N)}_{E'} | \psi^{(N)}_E \rangle = A(E')A(E) \frac{\sin \left( \frac{E' - E}{2} \log N + \varphi(E') - \varphi(E) \right)}{E' - E}, \quad N \to \infty. \]  

(38)

The limit \( N \to \infty \) is well defined here. In general, this is the Fresnel integral limit [5],

\[ \lim_{K \to \infty} \frac{\sin(Kx)}{x} = \pi \delta(x). \]  

(39)
Therefore, at $N \to \infty$ one obtains

$$\langle \psi_E | \psi_E \rangle = \pi A(E)^2 \delta(E - E') .$$  

(40)

In fact, this is the main result of our paper. Numerical analysis also shows that the spectrum is unbounded since

$$A(E) = A(-E) .$$  

(41)

5. Conclusion and Discussion

In this paper we have considered the stationary Schrödinger equation for the self-conjugated Hamiltonian $H = \frac{1}{i}(e + f)$, where $e$ and $f$ are creating and annihilation operators for the algebra $sl_2$ considered for the infinite-dimensional representation with lowest weight equals 1, equivalent to the usual Fock Space.

The eigenvector equation for operator $H$ is the the second order recursion equation. In this paper we have given detailed analysis for a solution of the recursion. General expression of $\psi_n(E)$ involves four functions: $A(E)$, $\psi(E)$, $\delta_n(E)$, $\varepsilon_n(E)$, see equation (22). We give the rigorous way to define $\delta(n, E)$ and $\varepsilon(n, E)$ analytically in the forms of series expansion with respect to $1/n$ and $E$, however the functions $A(E)$ and $\psi(E)$ are defined only numerically for real $E$.

The further development of the problem implies two ways: the first way is the further analysis of equation (11) in order to find analytical expressions for the asymptotic analytical functions $A(E)$ and $\varphi(E)$. The second way could be $q \neq 1$ generalisation of the problem. A preliminary analysis shows that $q \neq 1$ case leads to several unexpected mathematical phenomena.

References


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