COEFFICIENT ESTIMATES FOR BI-CONCAVE FUNCTIONS OF SAKAGUCHI TYPE

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Abstract. In this study, a new class \( \mathcal{CS}_{st}^{pq}(s,t,\alpha) \) of analytic and bi-concave functions with Sakaguchi type in the open unit disc were presented. The estimates on the first two Taylor-Maclaurin coefficients \( |a_2| \) and \( |a_3| \) were found for functions belonging to this class.

2010 Mathematics Subject Classification: Primary 30C45, secondary 30C50.

Keywords: Analytic function, Sakaguchi function, bi-univalent function, concave function.

1. Introduction, Preliminaries and Definition

Let \( \mathbb{C} \), \( \overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\} \) and \( \mathbb{R} \) denote the set of complex numbers, the extended complex plain and the set of real numbers respectively. Let \( \mathbb{D} \) denote the open unit disk. Let \( \mathcal{A} \) indicate the class of analytic functions in \( \mathbb{D} = \{z \in \mathbb{C} : |z| < 1\} \) given by

\[
f(z) = z + \sum_{n=2}^{\infty} a_n z^n,
\]

normalized by the condition \( f(0) = 0 = f'(0) - 1 \). Let \( \mathcal{S} \) be the set of all normalized analytic functions in \( \mathcal{A} \) which are univalent in \( \mathbb{D} \).

A univalent function \( f : \mathbb{D} \to \overline{\mathbb{C}} \) is called concave when \( f(\mathbb{D}) \) is concave, i.e. \( \overline{\mathbb{C}} \setminus f(\mathbb{D}) \) is convex. Concave univalent functions have already been studied in detail by several authors (see [1, 2, 3, 4, 7]).

A function \( f : \mathbb{D} \to \mathbb{C} \) is called a member of concave univalent functions with an opening angle \( \pi \alpha \) at infinity for \( \alpha \in (1, 2] \) if \( f \) satisfies the conditions given below:

1. \( f \) is analytic in \( \mathbb{D} \) which has normalization condition \( f(0) = 0 = f'(0) - 1 \). Additionaly, \( f(1) = \infty \).
2. \( f \) maps \( \mathbb{D} \) conformally onto a set whose complement is convex with respect to \( \mathbb{C} \).

3. The opening angle of \( f(\mathbb{D}) \) at infinity is less than or equal to \( \pi \alpha \), \( \alpha \in (1, 2] \).

Let us denote the class of concave univalent functions of order \( \beta \) by \( C_\beta(\alpha) \).

The analytic characterization for functions in \( C_\beta(\alpha) \) are as follows: For \( \alpha \in (1, 2] \) and \( \beta \in [0, 1) \), \( f \in C_\beta(\alpha) \) if and only if
\[
\Re P_f(z) > \beta, \quad \forall z \in \mathbb{D},
\]
for
\[
P_f(z) = \frac{2}{\alpha - 1} \left[ \frac{\alpha + 1 + z}{2} \frac{z f''(z)}{f'(z)} \right] - 1 - \frac{zf''(z)}{f'(z)}
\]
and \( f(0) = 0 = f'(0) - 1 \).

Also each \( f \in C_\beta(\alpha) \) has the Taylor expansion given by (1). Especially, for \( \beta = 0 \), we can obtain the class of concave univalent functions \( C_0(\alpha) \) which was studied in [2]. The closed set \( \overline{\mathbb{C}} \setminus f(\mathbb{D}) \) is convex and unbounded for \( f \in C_0(\alpha), \alpha \in (1, 2] \).

Now we define the class of concave functions with Sakaguchi type and order \( \beta \) by \( CS_\beta(s, t, \alpha) \) as follows:

For \( \alpha \in (1, 2], \beta \in [0, 1), s, t \in \mathbb{C} \) with \( s \neq t \), \( |t| \leq 1 \), \( f \in CS_\beta(s, t, \alpha) \) if and only if
\[
\Re P_f(z) > \beta, \quad \forall z \in \mathbb{D},
\]
for
\[
P_f(z) = \frac{2}{\alpha - 1} \left[ \frac{\alpha + 1 + z}{2} \frac{zf'(z)}{f'(z)} \right] - \frac{(s - t)(zf'(z))'}{(f(sz) - f(tz))'}.
\]

It is obvious that \( CS_\beta(1, 0, \alpha) \equiv C_\beta(\alpha) \).

For all \( f \in \mathcal{S} \), the Koebe 1/4 theorem [8] confirms that the image of \( \mathbb{D} \) under each univalent function \( f \in \mathcal{S} \) covers a disk of radius 1/4. Hence, each \( f \in \mathcal{A} \) has an inverse \( f^{-1} \), described by
\[f^{-1}(f(z)) = z \quad (z \in \mathbb{D})\]
and
\[f(f^{-1}(w)) = w \quad \left( |w| < r_0(f); r_0(f) \geq \frac{1}{4} \right).
\]
If $f$ is univalent and $g = f^{-1}$ is univalent in $\mathbb{D}$, the function $f \in \mathcal{A}$ is known to be bi-univalent in $\mathbb{D}$. If $f$ given by (1) is bi-univalent, then $g = f^{-1}$ can be arranged in the form of Taylor expansion given by
\[ g(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 - \cdots. \] (4)

Also, a function $f$ is bi-concave if both $f$ and $f^{-1}$ are concave.

Let us denote $\Sigma$ the class of all bi-univalent functions in $\mathbb{D}$. Lewin [10] investigated the class $\Sigma$ and showed that $|a_2| < 1.51$ for the function $f(z) \in \Sigma$. Also, several researchers obtained the coefficient boundaries for $|a_2|$ and $|a_3|$ of bi-univalent functions for some subclasses of the class $\Sigma$ in [9, 12, 13]. In addition, certain subclasses of bi-univalent functions, and also univalent functions consisting of strongly starlike, starlike and convex functions were studied by Brannan and Taha [5]. Some properties of bi-convex, bi-univalent and bi-starlike function classes have already been investigated by Brannan and Taha [5]. Furthermore, estimations for $|a_2|$ and $|a_3|$ were found by Bulut [6] for bi-starlike functions. The class of bi-concave functions was studied by Sakar and Güney in [11].

Now, we define the definition of bi-concave functions of Sakaguchi type as follows:

**Definition 1.** The function $f$ in (1) is called $\sum_{CS_{\beta}(s,t,\alpha)}$ if the conditions given below are satisfied: $f \in \Sigma$ and
\[
\text{Re} \left\{ \frac{2}{\alpha - 1} \left[ \frac{\alpha + 1}{2} \frac{1 + z}{1 - z} - \frac{(s - t)(zf'(z))'}{(f(sz) - f(tz))'} \right] \right\} > \beta, \quad z \in \mathbb{D} \text{ and } 0 \leq \beta < 1 \quad (5)
\]
and
\[
\text{Re} \left\{ \frac{2}{\alpha - 1} \left[ \frac{\alpha + 1}{2} \frac{1 - w}{1 + w} - \frac{(s - t)(wg'(w))'}{(g(sw) - g(tw))'} \right] \right\} > \beta, \quad w \in \mathbb{D} \text{ and } 0 \leq \beta < 1. \quad (6)
\]

where $g$ is given by (4) and $s, t \in \mathbb{C}$ with $s \neq t, |t| \leq 1$. In other words, $\sum_{CS_{\beta}(s,t,\alpha)}$ is the class of bi-concave functions of Sakaguchi type and order $\beta$.

It is obvious that $\sum_{CS_{\beta}(1,0,\alpha)} \equiv \sum_{C_{\beta}(\alpha)}$ (see [11]).

We next define the following subclass of $\mathcal{A}$, analogous to the definition given by Xu et al. [14].

**Definition 2.** Let us define the functions $p, q : \mathbb{D} \to \mathbb{C}$ satisfying the following condition
\[
\min \{ \text{Re}(p(z)), \text{Re}(q(z)) \} > 0 \quad (z \in \mathbb{D}) \text{ and } p(0) = q(0) = 1.
\]

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Also let the function \( f \), defined by (1.1), be in \( \mathcal{A} \). Then \( f \in \mathcal{CS}_p^\Sigma(s,t,\alpha) \) if the following conditions are satisfied:
\[
\frac{2}{\alpha - 1} \left[ \frac{\alpha + 1}{2} \frac{1 + z}{1 - z} - \frac{(s - t)(szf'(z))'}{f(sz) - f(tz)} \right] \in p(D), \quad (z \in D) \tag{7}
\]
and
\[
\frac{2}{\alpha - 1} \left[ \frac{\alpha + 1}{2} \frac{1}{1 + w} - \frac{(s - t)(wg'(w))'}{g(sw) - g(tw)} \right] \in q(D), \quad (w \in D) \tag{8}
\]
where the \( g \) is given in (4) and \( s,t \in \mathbb{C} \) with \( s \neq t, |t| \leq 1 \).

Remark. If we let
\[
p(z) = \frac{1 + (1 - 2\beta)z}{1 - z} \quad \text{and} \quad q(z) = \frac{1 - (1 - 2\beta)z}{1 + z}, \quad (0 \leq \beta < 1, z \in D) \tag{9}
\]
in the class \( \mathcal{CS}_p^\Sigma(s,t,\alpha) \) then we have \( \sum \mathcal{CS}_\beta(s,t,\alpha) \).

The aim of this paper is to estimate the initial coefficients for the bi-concave functions of Sakaguchi type in \( D \).

\section*{2. Initial Coefficient Boundary for \(|a_2| \) and \(|a_3| \)}

The estimations of initial coefficients for the class \( \mathcal{CS}_p^\Sigma(s,t,\alpha) \) of bi-concave functions of Sakaguchi type are presented in this section.

\textbf{Theorem 1.} If the function \( f(z) \) given by (1) is in \( \mathcal{CS}_p^\Sigma(s,t,\alpha) \) then
\[
|a_2| \leq \min \left\{ \frac{1}{|4 - 2u_2|^2} \left\{ (\alpha + 1)^2 + \frac{(\alpha^2 - 1)^2}{2} \left[ |p'(0)| + |q'(0)| \right] + \frac{8(\alpha - 1)^2}{8} \left[ |p'(0)|^2 + |q'(0)|^2 \right] \right\} \right\} \tag{10}
\]
and
\[
|a_3| \leq \min \left\{ \frac{(\alpha + 1)^2}{|4 - 2u_2|^2} + \frac{(\alpha - 1)}{8|9 - 3u_3|} \left[ |p''(0)| + |q''(0)| \right] \right\} \tag{11}
\]
and
\[
+ \frac{(\alpha^2 - 1)^2}{2|4 - 2u_2|^2} \left[ |p'(0)| + |q'(0)| \right] \right] + \frac{(\alpha - 1)^2}{8|4 - 2u_2|^2} \left[ |p'(0)|^2 + |q'(0)|^2 \right] \right\} \right\} \tag{12}
\]
and
\[
+ \frac{4}{|4(9 - 3u_3) - 8u_2(3 - 2u_2)|} \times \right\} \right\} \tag{13}
\]

\begin{align*}
|a_2| & \leq \min \left\{ \frac{1}{|4 - 2u_2|^2} \left\{ (\alpha + 1)^2 + \frac{(\alpha^2 - 1)^2}{2} |p'(0)| + |q'(0)| \right\} \right\} \tag{10} \\
|a_3| & \leq \min \left\{ \frac{(\alpha + 1)^2}{|4 - 2u_2|^2} + \frac{(\alpha - 1)}{8|9 - 3u_3|} \left[ |p''(0)| + |q''(0)| \right] \right\} \tag{11} \\
& \quad + \frac{(\alpha^2 - 1)^2}{2|4 - 2u_2|^2} \left[ |p'(0)| + |q'(0)| \right] \right] + \frac{(\alpha - 1)^2}{8|4 - 2u_2|^2} \left[ |p'(0)|^2 + |q'(0)|^2 \right] \right\} \right\} \tag{12} \\
& \quad + \frac{4}{|4(9 - 3u_3) - 8u_2(3 - 2u_2)|} \times \right\} \right\} \tag{13}
\end{align*}
where

\[ u_n = \sum_{k=1}^{n} s^{n-k} t^{k-1}, \quad s, t \in \mathbb{C} \text{ with } s \neq t, \ |t| \leq 1. \]

**Proof.** Firstly, we can write the argument inequalities in 7 and 8 in their equivalent forms as follows:

\[
\frac{2}{\alpha - 1} \left[ \frac{\alpha + 1 + z}{2} - \frac{(s-t)(z^f(z))'}{(f(sz) - f(tz))'} \right] = p(z) \quad (z \in \mathbb{D}),
\]

and

\[
\frac{2}{\alpha - 1} \left[ \frac{\alpha + 1 - w}{2} - \frac{(s-t)(w^g(w))'}{(g(sw) - g(tw))'} \right] = q(w) \quad (w \in \mathbb{D}).
\]

In addition, \(p(z)\) and \(q(w)\) can be expanded to Taylor-Maclaurin series as given below respectively

\[ p(z) = 1 + p_1 z + p_2 z^2 + \ldots \]

and

\[ q(w) = 1 + q_1 w + q_2 w^2 + \ldots \]

Now upon equating the coefficients of \(\frac{2}{\alpha - 1} \left[ \frac{\alpha + 1 + z}{2} - \frac{(s-t)(z^f(z))'}{(f(sz) - f(tz))'} \right]\) with those of \(p(z)\) and the coefficients of \(\frac{2}{\alpha - 1} \left[ \frac{\alpha + 1 - w}{2} - \frac{(s-t)(w^g(w))'}{(g(sw) - g(tw))'} \right]\) with those of \(q(w)\), we can write \(p(z)\) and \(q(w)\) as follows.

\[
p(z) = \frac{2}{\alpha - 1} \left[ \frac{\alpha + 1 + z}{2} - \frac{(s-t)(z^f(z))'}{(f(sz) - f(tz))'} \right] = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \ldots \quad (14)
\]

and

\[
q(w) = \frac{2}{\alpha - 1} \left[ \frac{\alpha + 1 - w}{2} - \frac{(s-t)(w^g(w))'}{(g(sw) - g(tw))'} \right] = 1 + q_1 w + q_2 w^2 + q_3 w^3 + \ldots \quad (15)
\]

Since

\[
\frac{(s-t)(z^f(z))'}{(f(sz) - f(tz))'} = \frac{1 + \sum_{n=2}^{\infty} n^2 a_n z^{n-1}}{1 + \sum_{n=2}^{\infty} n a_n u_n z^{n-1}} = 1 + [4 - 2u_2]a_2 z + ([9 - 3u_3]a_3 - 2u_2[4 - 2u_2]a_2^2) z^2 + \ldots
\]

where \(u_n = \sum_{k=1}^{n} s^{n-k} t^{k-1}\) and \(\frac{1+z}{1-z} = 1 + 2 \sum_{n=1}^{\infty} z^n = 1 + 2z + 2z^2 + 2z^3 + \ldots\) we obtain that

\[
\frac{2}{\alpha - 1} \left[ \frac{\alpha + 1 + z}{2} - \frac{(s-t)(z^f(z))'}{(f(sz) - f(tz))'} \right]
\]
From (4) and (6), we have

\[
1 + \frac{2((\alpha + 1) - [9 - 3u_3]a_3 + 2u_2[4 - 2u_2]a_3^2]}{(\alpha - 1)} = \sum_{n=1}^{\infty} u_n [9 - 3u_3]\]

Then

\[
p_1 = \frac{2[(\alpha + 1) - 4 - 2u_2]a_2]}{\alpha - 1}
\]

and

\[
p_2 = \frac{2[(\alpha + 1) - [9 - 3u_3]a_3 + 2u_2[4 - 2u_2]a_3^2]}{\alpha - 1}
\]

From (4) and (6), we have

\[
\left(\frac{g(sw) - g(tw)}{(sw)^{g'(w)}}\right)' = \frac{1 - 4a_2w + 9(2a_2^2 - a_3)w^2 + \ldots}{1 - 2uw_2 + 3u_3(2a_2^2 - a_3)w^2}
\]

\[
= 1 + \frac{[2u_2 - 4]a_2w + (9 - 3u_3)(2a_2^2 - a_3) + 2u_2(2u_2 - 4)a_3^2}{w^2 + \ldots}
\]

where \( u_n = \sum_{k=1}^{n} s^{n-k}k^{-1} \) and we know \( \frac{1}{1+w} = 1 + 2\sum_{n=1}^{\infty} (-1)^nw^n = 1 - 2w + 2w^2 - 2w^3 + \ldots \) Then from \( q(w) \) given by (15), we have

\[
\frac{2}{\alpha - 1} \left[ \frac{(\alpha + 1) - w}{1 + w} \right] = \frac{2}{\alpha - 1} \left[ \frac{(\alpha + 1) - (\alpha + 1)w + (\alpha + 1)w^2 - \ldots}{w} \right]
\]

\[
-1 - \frac{[2u_2 - 4]a_2w + (9 - 3u_3)(2a_2^2 - a_3) + 2u_2(2u_2 - 4)a_3^2}{w^2 + \ldots}
\]

\[
= 1 - \frac{2[(\alpha + 1) + [2u_2 - 4]a_2]}{(\alpha - 1)} + \frac{2[(\alpha + 1) - [9 - 3u_3](2a_2^2 - a_3) + 2u_2(2u_2 - 4)a_3^2]}{(\alpha - 1)} \]

So we can obtain \( q_1 \) and \( q_2 \) as follows

\[
q_1 = -\frac{2[(\alpha + 1) + [2u_2 - 4]a_2]}{(\alpha - 1)}
\]

\[
q_2 = \frac{2[(\alpha + 1) - [(9 - 3u_3)(2a_2^2 - a_3) + 2u_2(2u_2 - 4)a_3^2]]}{(\alpha - 1)}
\]

From (16) and (18) we obtain

\[
p_1 = -q_1
\]

and

\[
a_2^2 = \frac{(\alpha + 1)^2}{[4 - 2u_2]^2} - \frac{(\alpha^2 - 1)}{2^2[4 - 2u_2]^2}p_1q_1 + \frac{(\alpha - 1)^2}{8[4 - 2u_2]^2}[p_1^2 + q_1^2]
\]

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or
\[ a^2_2 = \frac{1}{[4 - 2u_2]^2} \left\{ (\alpha + 1)^2 - \frac{(\alpha^2 - 1)}{2}[p_1 - q_1] + \frac{(\alpha - 1)^2}{8}[p_1^2 + q_1^2] \right\}. \] (21)

Also, from (17) and (19) we obtain that
\[ a^2_2 = \frac{(1 - \alpha)}{[4(9 - 3u_3) - 8u_2(4 - 2u_2)]}[p_2 + q_2] + \frac{4(\alpha + 1)}{[4(9 - 3u_3) - 8u_2(4 - 2u_2)]} \]
or
\[ a^2_2 = \frac{1}{[4(9 - 3u_3) - 8u_2(4 - 2u_2)]} \left\{ (1 - \alpha)[p_2 + q_2] + 4(\alpha + 1) \right\}. \] (22)

Therefore, we find from (21) and (22)
\[ |a_2|^2 = \frac{1}{|[4 - 2u_2]|^2} \left\{ (\alpha + 1)^2 + \frac{(\alpha^2 - 1)}{2}|p'(0)| + |q'(0)| \right\} + \frac{(\alpha - 1)^2}{8}(|p'(0)|^2 + |q'(0)|^2). \]
and
\[ |a_2|^2 = \frac{1}{|4(9 - 3u_3) - 8u_2(4 - 2u_2)|} \left\{ \frac{(\alpha - 1)}{2}|p''(0)| + |q''(0)| + 4(\alpha + 1) \right\}. \]

So we obtain the upper bound of \( |a_2| \) as stated in (10).

Now, to obtain the upper bound for the coefficient \( |a_3| \) we use (17) and (19). So we obtain
\[ (\alpha - 1)(p_2 - q_2) = 4[9 - 3u_3]a^2_2 - 4[9 - 3u_3]a_3. \]

From (21), we find
\[ 4[9 - 3u_3]a_3 = - (\alpha - 1)(p_2 - q_2) + \frac{4[9 - 3u_3]}{[4(9 - 3u_3) - 8u_2(4 - 2u_2)]} \left\{ (1 - \alpha)[p_2 + q_2] + 4(\alpha + 1) \right\} \]
or
\[ a_3 = - \frac{(\alpha - 1)}{4[9 - 3u_3]}(p_2 - q_2) + \frac{1}{[4(9 - 3u_3) - 8u_2(4 - 2u_2)]} \left\{ (1 - \alpha)[p_2 + q_2] + 4(\alpha + 1) \right\} \]
\[ \Rightarrow a_3 = \frac{4(\alpha + 1)}{[4(9 - 3u_3) - 8u_2(4 - 2u_2)]} \]

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Also, we obtain from (22)

\[ |a_3| \leq \frac{4}{|4(9-3u_3) - 8u_2(4-2u_2)|} \left[ (\alpha + 1) + \frac{(\alpha - 1)}{4|9-3u_3|} [(9-3u_3) - u_2(4-2u_2)] |p''(0)| + |u_2(4-2u_2)| |q''(0)| \right]. \]  

(23)

We thus find that

\[ |a_3| \leq \frac{4}{|4(9-3u_3) - 8u_2(4-2u_2)|} \times \left[ (\alpha + 1) + \frac{(\alpha - 1)}{4|9-3u_3|} [(9-3u_3) - u_2(4-2u_2)] |p''(0)| + |u_2(4-2u_2)| |q''(0)| \right]. \]

Also, we obtain from (22)

\[ 4|9-3u_3|a_3 = -(\alpha - 1)(p_2 - q_2) + 4|9-3u_3| \left\{ \frac{(\alpha + 1)^2}{2[4-2u_2]^2} - \frac{(\alpha - 1)^2}{2[4-2u_2]^2} (p_2 - q_2) - \frac{(\alpha - 1)^2}{4(9-3u_3)} (p_2 - q_2) - \frac{(\alpha - 1)^2}{8[4-2u_2]^2} (p_2 - q_2)^2 \right\}. \]

\[ \Rightarrow a_3 = \frac{(\alpha + 1)^2}{4[4-2u_2]^2} - \frac{(\alpha - 1)^2}{4|9-3u_3|} (p_2 - q_2) - \frac{(\alpha - 1)^2}{2[4-2u_2]^2} (p_2 - q_2) + \frac{(\alpha - 1)^2}{8|4-2u_2|^2} (p_2 - q_2)^2. \]  

(24)

We thus find that

\[ |a_3| \leq \frac{(\alpha + 1)^2}{4[4-2u_2]^2} + \frac{(\alpha - 1)^2}{8|9-3u_3|} (|p''(0)| + |q''(0)|) + \frac{(\alpha - 1)^2}{2[4-2u_2]^2} (|p'(0)| + |q'(0)|) + \frac{(\alpha - 1)^2}{8|4-2u_2|^2} (|p'(0)| + |q'(0)|)^2. \]

So, the proof of Theorem 1 is completed.

If we set

\[ p(z) = \frac{1 + (1 - 2\beta)z}{1 - z} \quad \text{and} \quad q(z) = \frac{1 - (1 - 2\beta)z}{1 + z} \quad (0 \leq \beta < 1, z \in \mathbb{D}) \]

in Theorem 1, we can obtain the following corollary.

**Corollary 1.** Let \( f \) given by (1) be in the class \( \sum_{C^*_{\beta}(s,t,\alpha)} \) \((0 \leq \beta < 1)\). Then

\[ |a_2| \leq \sqrt{\frac{4 \{(\alpha - 1)(1 - \beta) + (\alpha + 1)\}}{|4(9-3u_3) - 8u_2(4-2u_2)|}}, \]

and

\[ |a_3| \leq \frac{4}{|4(9-3u_3) - 8u_2(4-2u_2)|} \times \left[ (\alpha + 1) + \frac{(\alpha - 1)}{|9-3u_3|} [(9-3u_3) - u_2(4-2u_2)] + |u_2(4-2u_2)| (1 - \beta) \right]. \]

where \( u_n = \sum_{k=1}^{n} s^{n-k-1} t^{k-1}, s, t \in \mathbb{C} \) with \( s \neq t, |t| \leq 1 \).

Last of all, if we take \( s = 1 \) and \( t = 0 \) in Theorem 1 and Corollary 1, we can obtain Theorem 2.1 and Corollary 3.1 in [11] respectively.
References


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