INCLUSION RELATIONSHIPS AND SOME INTEGRAL-PRESERVING PROPERTIES OF CERTAIN CLASSES OF MEROMORPHIC P-VALENT FUNCTIONS

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Abstract. We introduce some integral operators defined on the space of p-valent meromorphic functions in the class Σ_p. By using these integral operators, we define several subclasses of p-valent meromorphic functions and investigate various inclusion relationship and integral-preserving properties.

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1. Introduction

Let Σ_p denotes the class of functions f given by

\[ f(z) = \frac{1}{z^p} + \sum_{k=1}^{\infty} a_{k-p} z^{k-p} \quad (p \in \mathbb{N} = \{1, 2, 3, \ldots\}) \quad (1) \]

which are analytic and p-valent in the punctured unit disc

\[ U^* = \{ z \in \mathbb{C} : 0 < |z| < 1 \}. \]

A function \( f \in \Sigma_p \) is said to be in the class \( \Sigma S^*_p(\alpha) \) of meromorphic p-valent starlike functions of order \( \alpha \) in \( U^* \) if

\[ \text{Re} \left( \frac{zf'(z)}{f(z)} \right) < -\alpha, \quad (z \in U^*; 0 \leq \alpha < p), \quad (2) \]

also, a function \( f \in \Sigma_p \) is said to be in the class \( \Sigma C_p(\alpha) \) of meromorphic p-valent convex functions of order \( \alpha \) in \( U^* \) if

\[ \text{Re} \left( 1 + \frac{zf''(z)}{f'(z)} \right) < -\alpha, \quad (z \in U^*; 0 \leq \alpha < p). \quad (3) \]
It is easy to observe from (2) and (3) that
\[ f \in \Sigma C_p(\alpha) \iff -\frac{zf'}{p} \in \Sigma S_p^*(\alpha). \tag{4} \]

A function \( f \in \Sigma_p \) is said to be in the class \( \Sigma K_p(\beta, \alpha) \) of meromorphic \( p \)-valent close-to-convex functions of order \( \beta \) and type \( \alpha \) in \( \mathbb{U}^* \) if there exist a function \( g \in \Sigma S_p^*(\alpha) \) such that
\[ \Re \left( \frac{zf'(z)}{g(z)} \right) < -\beta, \quad (z \in \mathbb{U}^*; 0 \leq \alpha, \beta < p), \tag{5} \]
furthermore, a function \( f \in \Sigma_p \) is said to be in the class \( \Sigma K_p^*(\beta, \alpha) \) of meromorphic \( p \)-valent quasi-convex functions of order \( \beta \) and type \( \alpha \) in \( \mathbb{U}^* \) if there exist a function \( g \in \Sigma C_p(\alpha) \) such that
\[ \Re \left( \frac{(zf'(z))'}{g'(z)} \right) < -\beta, \quad (z \in \mathbb{U}^*; 0 \leq \alpha, \beta < p). \tag{6} \]

It is easy to observe from (5) and (6) that
\[ f \in \Sigma K_p^*(\beta, \alpha) \iff -\frac{zf'}{p} \in \Sigma K_p(\beta, \alpha). \tag{7} \]

**Definition 1.** Let \( 0 \leq \mu \leq 1; 0 \leq \gamma \leq 1; p \in \mathbb{N} \) and \( f \in \Sigma_p \), we introduce the \( p \)-valent Rafid operator \( S_{\gamma, \mu}^\gamma : \Sigma_p \to \Sigma_p \) which is defined by
\[ S_{\gamma, \mu}^\gamma f(z) = \frac{1}{(1 - \mu)\gamma + 1} \frac{\Gamma(\gamma + 1)}{\Gamma(\gamma)} \int_0^\infty t^{\gamma + p} e^{-\frac{t}{1 - \mu}} f(zt)dt \tag{8} \]
then,
\[ S_{\gamma, \mu}^\gamma f(z) = \frac{1}{z^p} + \sum_{k=1}^\infty L(\gamma, \mu, k) a_{k-p} z^{k-p} \tag{9} \]
where,
\[ L(\gamma, \mu, k) = (1 - \mu)^k (\gamma + 1)_k \]
and \( (\nu)_k \) denotes the Pochhammer symbol given by
\[ (\nu)_k = \frac{\Gamma(\nu + k)}{\Gamma(\nu)} = \begin{cases} 1 & \text{if } k = 0, \\ \nu(\nu + 1)\cdots(\nu + k - 1) & \text{if } k \in \mathbb{N}. \end{cases} \tag{10} \]
Remark 1. Putting \( p = 1 \) in (8) we have the Rafid operator \( S_{\mu}^{\gamma} \) which is introduced by Rosy and Varma [4].

Remark 2. Using the equation (9), it is easy to see that

\[
S_{\mu,p}^{\gamma} \left( z f'(z) \right) = z \left( S_{\mu,p}^{\gamma} f(z) \right)'
\]

and,

\[
z \left( S_{\mu,p}^{\gamma} f(z) \right)' = (\gamma + 1) S_{\mu,p}^{\gamma+1} f(z) - (\gamma + p + 1) S_{\mu,p}^{\gamma} f(z)
\]

By putting \( a_{k-p} = 1, \forall k \) in (9), we get

\[
\psi_{\mu,p}^{\gamma}(z) = \frac{1}{z^p} + \sum_{k=1}^{\infty} L(\gamma, \mu, k) z^{k-p}
\]

and \( \varphi_{\mu,p}^{\gamma,\lambda}(z) \) be defined using the Hadmard product as

\[
\varphi_{\mu,p}^{\gamma,\lambda}(z) = \frac{1}{z^p(1-z)^\lambda}
\]

therefore,

\[
\varphi_{\mu,p}^{\gamma,\lambda}(z) = \frac{1}{z^p} + \sum_{k=1}^{\infty} \frac{(\lambda)_k}{(1-\mu)^k (\gamma+1)_k} z^{k-p}
\]

Definition 2. For \( 0 \leq \mu \leq 1, 0 \leq \gamma \leq 1, \lambda > 0 \) and \( p \in \mathbb{N} \), we introduce the integral operator \( J_{\mu,p}^{\gamma,\lambda} : \Sigma_p \to \Sigma_p \) which is defined by

\[
J_{\mu,p}^{\gamma,\lambda} f(z) = \varphi_{\mu,p}^{\gamma,\lambda}(z) \ast f(z)
\]

Therefore,

\[
J_{\mu,p}^{\gamma,\lambda} f(z) = \frac{1}{z^p} + \sum_{k=1}^{\infty} \frac{1}{(1-\mu)^k (\gamma+1)_k (1)_k} a_{k-p} z^{k-p}
\]

Remark 3. Using equation (15), it is easy to see that

\[
z \left( J_{\mu,p}^{\gamma+1,\lambda} f(z) \right)' = (\gamma + 1) J_{\mu,p}^{\gamma,\lambda} f(z) - (p + \gamma + 1) J_{\mu,p}^{\gamma+1,\lambda} f(z),
\]

and

\[
z \left( J_{\mu,p}^{\gamma,\lambda} f(z) \right)' = \lambda J_{\mu,p}^{\gamma,\lambda+1} f(z) - (p + \lambda) J_{\mu,p}^{\gamma,\lambda} f(z).
\]
We now define the following subclasses of the meromorphic function class $\Sigma_p$ by means of the integral operator $J_{\mu,p}^{\gamma,\lambda}$ given by (14).

$$
\Sigma S_{\mu,p}^{\gamma,\lambda}(\alpha) = \left\{ f : f \in \Sigma_p \text{ and } J_{\mu,p}^{\gamma,\lambda}f(z) \in \Sigma S_p^{\alpha}(\alpha) \right\} \quad (18)
$$

$$
\Sigma C_{\mu,p}^{\gamma,\lambda}(\alpha) = \left\{ f : f \in \Sigma_p \text{ and } J_{\mu,p}^{\gamma,\lambda}f(z) \in \Sigma C_p(\alpha) \right\} \quad (19)
$$

$$
\Sigma K_{\mu,p}^{\gamma,\lambda}(\beta,\alpha) = \left\{ f : f \in \Sigma_p \text{ and } J_{\mu,p}^{\gamma,\lambda}f(z) \in \Sigma K_p^{\beta}(\alpha) \right\} \quad (20)
$$

$$
\Sigma K_{\mu,p}^{\gamma,\lambda}(\beta,\alpha) = \left\{ f : f \in \Sigma_p \text{ and } J_{\mu,p}^{\gamma,\lambda}f(z) \in \Sigma K_p^{\beta}(\alpha) \right\} \quad (21)
$$

where

$$
z \in U, 0 \leq \alpha < p, p \in \mathbb{N}.
$$

Before we establish our main result, we need the following lemma due to Miller and Mocanu [3].

**Lemma 1.** Let $\theta(u,v)$ be a complex-valued function such that $\theta : D \to \mathbb{C}$, $D \subset \mathbb{C} \times \mathbb{C}$ ($\mathbb{C}$ is the complex plane) and let $u = u_1 + iu_2$ and $v = v_1 + iv_2$. Suppose that $\theta(u,v)$ satisfies the following conditions:

- $\theta(u,v)$ is continuous in $D$;
- $(1,0) \in D$ and $\text{Re}\{\theta(1,0)\} > 0$;
- for all $(iu_2,v_1) \in D$ such that $v_1 \leq \frac{-1}{2}(1 + u_2^2)$, $\text{Re}\{\theta(iu_2,v_1)\} \leq 0$.

Let,

$$
q(z) = 1 + q_1z + q_2z^2 + ... \quad (22)
$$

be an analytic in $U$ such that $(q(z), q'(z)) \in D$ ($z \in U$). If $\text{Re}\{\theta(q(z), q'(z))\} > 0$, then $\text{Re}\{q(z)\} > 0$.

2. Inclusion Relationships

In this section, we give several inclusion relationships for $p$-valent meromorphic function classes, which are associated with the integral operator $J_{\mu,p}^{\gamma,\lambda}$.

**Theorem 2.** Let $0 \leq \mu \leq 1$, $0 \leq \gamma \leq 1$, $\lambda > 0$ and $0 \leq \alpha < p, p \in \mathbb{N}$, then

$$
\Sigma S_{\mu,p}^{\gamma,\lambda+1}(\alpha) \subset \Sigma S_{\mu,p}^{\gamma,\lambda}(\alpha) \subset \Sigma S_{\mu,p}^{\gamma+1,\lambda}(\alpha) \quad (23)
$$
Proof. (i) We first show that

\[ \Sigma_{\mu,p} S^{*\gamma,\lambda+1}(\alpha) \subset \Sigma_{\mu,p} S^{*\gamma,\lambda}(\alpha) \]  (24)

Let \( f(z) \in \Sigma_{\mu,p} S^{*\gamma,\lambda+1}(\alpha) \) and set

\[ z \left( J_{\mu,p}^\gamma f(z) \right)' = -\alpha - (p - \alpha)q(z) \]  (25)

where \( q(z) \) is given by (22). By using equation (17), we have

\[ \frac{\lambda J_{\mu,p}^{\gamma,\lambda+1} f(z)}{J_{\mu,p}^{\gamma,\lambda} f(z)} = (p + \lambda - \alpha) - (p - \alpha)q(z) \]  (26)

Differentiating (25) logarithmically with respect to \( z \), we obtain

\[ \frac{z(J_{\mu,p}^{\gamma,\lambda+1} f(z))'}{J_{\mu,p}^{\gamma,\lambda+1} f(z)} = \frac{z(J_{\mu,p}^{\gamma,\lambda} f(z))'}{J_{\mu,p}^{\gamma,\lambda} f(z)} + \frac{z(p - \alpha)q'(z)}{(p - \alpha)q(z) - (p + \lambda - \alpha)} \]

= \(-\alpha - (p - \alpha)q(z) + \frac{z(p - \alpha)q'(z)}{(p - \alpha)q(z) - (p + \lambda - \alpha)} \)

Let now,

\[ \theta(u, v) = (p - \alpha)u - \frac{(p - \alpha)v}{(p - \alpha)u - (p + \lambda - \alpha)} \]  (27)

where \( u = q(z) = u_1 + iu_2 \) and \( v = zq'(z) = v_1 + iv_2 \). Then,

- \( \theta(u, v) \) is continuous in \( D = \{ \mathbb{C} \setminus \left( \frac{p+\lambda-\alpha}{p-\alpha} \right) \} \times \mathbb{C} \);
- \( (1, 0) \in D \) with \( \text{Re} \{ \theta(1, 0) \} = p - \alpha > 0 \);
- for all \( (iu_2, v_1) \in D \) such that \( v_1 \leq \frac{1}{2}(1 + u_2^2) \), we have

\[ \text{Re} \{ \theta(iu_2, v_1) \} = \text{Re} \left\{ \frac{(p - \alpha)iu_2 - (p - \alpha)v_1}{(p - \alpha)iu_2 - (p + \lambda - \alpha)} \right\} \]

\[ = \text{Re} \left\{ \frac{- (p - \alpha)v_1}{(p - \alpha)iu_2 - (p + \lambda - \alpha)} \ast \frac{- (p - \alpha)iu_2 - (p + \lambda - \alpha)}{-(p - \alpha)iu_2 - (p + \lambda - \alpha)} \right\} \]

\[ = \frac{(p + \lambda - \alpha)(p - \alpha)v_1}{((p - \alpha)u_2)^2 + (p + \lambda - \alpha)^2} \]

\[ \leq -\frac{(p + \lambda - \alpha)(p - \alpha)(1 + u_2^2)}{2 \left[ ((p - \alpha)u_2)^2 + (p + \lambda - \alpha)^2 \right]} < 0 \]
which shows that $\theta(u,v)$ satisfies the hypotheses of Lemma 1 then $\text{Re} q(z) > 0$. Consequently, we easily obtain the inclusion relationship (24).

(ii) by using the similar argument in proving relation (24) together with (16) and $\theta(u,v)$ is continuous in $D = \left\{ \mathbb{C} \setminus \left( \frac{p+\gamma+1-\alpha}{p-\alpha} \right) \right\} \times \mathbb{C}$, we can prove the right part of Theorem 1 that is

$$\Sigma S^{\gamma+1,\lambda}_{\mu,p}(\alpha) \subset \Sigma S^{\gamma,\lambda}_{\mu,p}(\alpha)$$  \hspace{1cm} (28)

By combining the inclusion relationships (24) and (28), we complete the proof of Theorem 1.

**Theorem 3.** Let $0 \leq \mu \leq 1$, $0 \leq \gamma \leq 1, \lambda > 0$ and $0 \leq \alpha < p, p \in \mathbb{N}$, then

$$\Sigma C^{\gamma,\lambda+1}_{\mu,p}(\alpha) \subset \Sigma C^{\gamma,\lambda}_{\mu,p}(\alpha) \subset \Sigma C^{\gamma+1,\lambda}_{\mu,p}(\alpha)$$  \hspace{1cm} (29)

**Proof.** Let $f(z) \in \Sigma C^{\gamma,\lambda+1}_{\mu,p}(\alpha)$. Then, from (18), we have

$$J^{\gamma,\lambda+1}_{\mu,p}f \in \Sigma C_p(\alpha)$$

Furthermore, in view of (4), we find that

$$-\frac{z}{p} \left( J^{\gamma,\lambda+1}_{\mu,p}f \right)' \in \Sigma S^*_p(\alpha)$$

that is,

$$J^{\gamma,\lambda+1}_{\mu,p} \left( -\frac{z}{p} f' \right) \in \Sigma S^*_p(\alpha)$$

Therefore,

$$-\frac{z}{p} f' \in \Sigma S^{\gamma,\lambda+1}_{\mu,p}$$

In view of Theorem 1, we have

$$-\frac{z}{p} f' \in \Sigma S^{\gamma,\lambda+1}_{\mu,p} \subset \Sigma S^{\gamma,\lambda}_{\mu,p}(\alpha)$$

Then, we get that $f \in \Sigma C^{\gamma,\lambda}_{\mu,p}(\alpha)$ which implies that,

$$\Sigma C^{\gamma,\lambda+1}_{\mu,p}(\alpha) \subset \Sigma C^{\gamma,\lambda}_{\mu,p}(\alpha)$$

The right part of Theorem 2 can be proved using the same arguments. The proof is thus completed.

**Theorem 4.** Let $0 \leq \mu \leq 1$, $0 \leq \gamma \leq 1$, $\lambda > 0$ and $0 \leq \alpha < p, p \in \mathbb{N}$, then

$$\Sigma K^{\gamma,\lambda+1}_{\mu,p}(\beta,\alpha) \subset \Sigma K^{\gamma,\lambda}_{\mu,p}(\beta,\alpha) \subset \Sigma K^{\gamma+1,\lambda}_{\mu,p}(\beta,\alpha)$$  \hspace{1cm} (30)
Proof. (i) let us first prove that
\[ \Sigma K_{\mu,p}^{\gamma,\lambda+1}(\beta, \alpha) \subset \Sigma K_{\mu,p}^{\gamma,\lambda}(\beta, \alpha) \] (31)

Let \( f(z) \in \Sigma K_{\mu,p}^{\gamma,\lambda+1}(\beta, \alpha) \). Then there exists a function \( \Omega(z) \in \Sigma S_p^{*}(\alpha) \) such that
\[
\text{Re} \left( \frac{z \left( J_{\mu,p}^{\gamma,\lambda+1} f(z) \right)'}{\Omega(z)} \right) < -\beta \quad (z \in U^*)
\]

We set
\[ \Omega(z) = J_{\mu,p}^{\gamma,\lambda+1} g(z) \]

So that we have
\[
g(z) \in \Sigma S_{\mu,p}^{*\gamma,\lambda+1}(\alpha) \quad \text{and} \quad \text{Re} \left( \frac{z \left( J_{\mu,p}^{\gamma,\lambda+1} f(z) \right)'}{J_{\mu,p}^{\gamma,\lambda+1} g(z)} \right) < -\beta \quad (z \in U^*)
\]

By setting that,
\[
\frac{z \left( J_{\mu,p}^{\gamma,\lambda} f(z) \right)'}{J_{\mu,p}^{\gamma,\lambda} g(z)} = -\beta - (p - \beta) q(z)
\] (32)

where \( q(z) \) is given by (22). Then, By using the identity (17), we obtain
\[
\frac{z \left( J_{\mu,p}^{\gamma,\lambda+1} f(z) \right)'}{J_{\mu,p}^{\gamma,\lambda+1} g(z)} = \frac{J_{\mu,p}^{\gamma,\lambda+1} (zf'(z))}{J_{\mu,p}^{\gamma,\lambda+1} g(z)} = z \left( J_{\mu,p}^{\gamma,\lambda} (zf'(z)) \right)' + (p + \lambda) J_{\mu,p}^{\gamma,\lambda} (zf'(z))
\]
\[
= z \left( J_{\mu,p}^{\gamma,\lambda} g(z) \right)' + (p + \lambda) J_{\mu,p}^{\gamma,\lambda} g(z)
\]
\[
= z \left( J_{\mu,p}^{\gamma,\lambda} g(z) \right)' + (p + \lambda) J_{\mu,p}^{\gamma,\lambda} g(z)
\]

Since \( g(z) \in \Sigma S_{\mu,p}^{*\gamma,\lambda+1}(\alpha) \), by Theorem 1, we can setting
\[
\frac{z \left( J_{\mu,p}^{\gamma,\lambda} g(z) \right)'}{J_{\mu,p}^{\gamma,\lambda} g(z)} = -\alpha - (p - \alpha) H(z)
\] (33)
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where $H(z) = g_1(x, y) + ig_2(x, y)$ and $\text{Re} \{H(z)\} = g_1(x, y) > 0 \ (z \in U^*)$. Then,

$$
\frac{z (J_{\mu, p}^\gamma \lambda + 1 f(z))'}{J_{\mu, p}^\gamma \lambda + 1 g(z)} = \frac{z (J_{\mu, p}^\gamma \lambda (zf'(z))')}{J_{\mu, p}^\gamma \lambda g(z)} + (p + \lambda) \left[ -\beta - (p - \beta)q(z) \right] - \alpha - (p - \alpha)H(z) + (p + \lambda) \tag{34}
$$

Thus we have from (32) that

$$
z \left( J_{\mu, p}^\gamma \lambda f(z) \right)' = -J_{\mu, p}^\gamma \lambda g(z) \left[ \beta + (p - \beta)q(z) \right] \tag{35}
$$

Differentiating both sides of (35) with respect to $z$, we obtain

$$
z \left( J_{\mu, p}^\gamma \lambda zf'(z) \right)' = -\beta - (p - \beta)q(z) \frac{z (J_{\mu, p}^\gamma \lambda g(z))'}{J_{\mu, p}^\gamma \lambda g(z)} - (p - \beta)zq'(z) \tag{36}
$$

$$
= -(p - \beta)zq'(z) + [\beta + (p - \beta)q(z)] [\alpha + (p - \alpha)H(z)]
$$

Now, substituting from (36) into (34), we have

$$
z \left( J_{\mu, p}^\gamma \lambda + 1 f(z) \right)' = \beta - (p - \beta)q(z) + \frac{(p - \beta)zq'(z)}{(p - \alpha)H(z) + \alpha - (p + \lambda)}
$$

Taking $u = q(z) = u_1 + iu_2$ and $v = zq'(z) = v_1 + iv_2$, we define the function $\Phi(u, v)$ by

$$
\Phi(u, v) = (p - \beta)u - \frac{(p - \beta)v}{(p - \alpha)H(z) + \alpha - (p + \lambda)} \tag{37}
$$

where $(u, v) \in \mathcal{D} = (\mathbb{C} \setminus D^*) \times \mathbb{C}$ and

$$
D^* = \left\{ z : z \in \mathbb{C} \text{ and } \text{Re} \{H(z)\} = g_1(x, y) \geq 1 + \frac{\lambda}{p - \alpha} \right\}
$$

Then, it follows from (37) that,

- $\Phi(u, v)$ is continuous in $\mathcal{D} = (\mathbb{C} \setminus D^*) \times \mathbb{C}$;
- $(1, 0) \in \mathcal{D}$ with $\text{Re} \{\Phi(1, 0)\} = p - \beta > 0$;
for all \((iu_2, v_1) \in \mathcal{D}\) such that \(v_1 \leq \frac{1}{2^2}(1 + u_2^2)\), we have

\[
\Re \{\Phi(iu_2, v_1)\} = \Re \left\{ \frac{(p - \beta)v_1}{(p - \alpha)H(z) + \alpha - (p + \lambda)} \right\}
\]

\[
= \Re \left\{ \frac{- (p - \beta)v_1}{i(p - \alpha)g_2(x, y) + [(p - \alpha)g_1(x, y) + \alpha - (p + \lambda)]} \right\}
\]

\[
= \frac{[\gamma (p - \alpha)g_2(x, y)]^2 + [(p - \alpha)g_1(x, y) + \alpha - (p + \lambda)]^2}{(p - \beta) [(p + \lambda) - \alpha - (p - \alpha)g_1(x, y)] (1 + u_2^2)} < 0
\]

which proves that \(\Phi(u, v)\) satisfies the hypotheses of Lemma 1, then \(\Re(q(z)) > 0\). Thus, in the light of (32), we easily deduce the inclusion relationship (31).

(ii) By using the similar argument in proving relation (31) together with (16) and

\[
D^* = \left\{ z : z \in \mathbb{C} \text{ and } \Re \{H(z)\} = g_1(x, y) \geq \frac{\gamma + 1 - \alpha}{p - \alpha} \right\}
\]

we can prove the right part of Theorem 3, that is

\[
\Sigma K_{\mu, p}^{\gamma, \lambda}(\beta, \alpha) \subset \Sigma K_{\mu, p}^{\gamma+1, \lambda}(\beta, \alpha)
\]

By combining the inclusion relationships (31) and (38), we complete the proof of Theorem 3.

Theorem 5. Let \(0 \leq \mu \leq 1\), \(0 \leq \gamma \leq 1\), \(\lambda > 0\) and \(0 \leq \alpha < p, p \in \mathbb{N}\), then

\[
\Sigma K_{\mu, p}^{\gamma, \lambda}(\beta, \alpha) \subset \Sigma K_{\mu, p}^{\gamma+1, \lambda}(\beta, \alpha) \subset \Sigma K_{\mu, p}^{\gamma+1, \lambda}(\beta, \alpha)
\]

Proof. Just as we derived Theorem 2 as a consequence of Theorem 1 by using the equivalence (4), we can also use the equivalence (7) to prove this Theorem as a consequence of Theorem 3.

3. A set of integral-preserving properties

In this section, we present some integral-preserving properties of the meromorphic function classes introduced here. We first recall a familiar integral operator \(L_{c,p}\) which introduced by Bernardi [2] defined by

\[
L_{c,p}f(z) = \frac{c}{z^{c+p}} \int_0^z t^{c+p-1} f(t) dt \quad (f \in \Sigma_p; c > 0; p \in \mathbb{N})
\]
which satisfies the following relationship:

\[
J_{\gamma,\lambda}^{\mu,p} \frac{(J_{\gamma,\lambda}^{\mu,p} \mathcal{L}_{c,p} f(z))^{'}}{J_{\gamma,\lambda}^{\mu,p} \mathcal{L}_{c,p} f(z)} = c J_{\gamma,\lambda}^{\mu,p} f(z) - (p + c) J_{\gamma,\lambda}^{\mu,p} \mathcal{L}_{c,p} f(z)
\]  

(41)

In order to obtain the integral-preserving properties involving the integral operator \(\mathcal{L}_{c,p}\), we also need the following lemma which is popularly known as Jack’s lemma [1].

**Lemma 6.** Let \(\omega(z)\) be a non-constant function analytic in \(U\) with \(\omega(0) = 0\). If \(|\omega(z)|\) attains its maximum value on the circle \(|z| = r < 1\) at \(z_0\), then

\[
z_0 \omega'(z_0) = \zeta \omega(z_0)
\]  

(42)

where \(\zeta\) is a real number and \(\zeta \geq 1\).

Unless otherwise mentioned, we assume in the reminder of this section that \(c, \lambda > 0; 0 \leq \mu, \gamma \leq 1; \zeta \geq 1\) and \(0 \leq \alpha, \beta < p, p \in \mathbb{N}\).

**Theorem 7.** If \(f(z) \in \Sigma_{\mu,p}^{\gamma,\lambda} (\alpha)\), Then

\[
\mathcal{L}_{c,p} f(z) \in \Sigma_{\mu,p}^{\gamma,\lambda} (\alpha).
\]  

(43)

**Proof.** Suppose that \(f(z) \in \Sigma_{\mu,p}^{\gamma,\lambda} (\alpha)\) and let

\[
\frac{z \left( J_{\mu,p}^{\gamma,\lambda} \mathcal{L}_{c,p} f(z) \right)'}{J_{\mu,p}^{\gamma,\lambda} \mathcal{L}_{c,p} f(z)} = -\frac{p + (p - 2\alpha) \omega(z)}{1 - \omega(z)}
\]  

(44)

where \(\omega(0) = 0\). Then, by using (41) and (44), we have

\[
\frac{J_{\mu,p}^{\gamma,\lambda} f(z)}{J_{\mu,p}^{\gamma,\lambda} \mathcal{L}_{c,p} f(z)} = \frac{c - (2p + c - 2\alpha) \omega(z)}{c (1 - \omega(z))}
\]  

(45)

which, upon logarithmic differentiation, we get

\[
\frac{z \left( J_{\mu,p}^{\gamma,\lambda} f(z) \right)'}{J_{\mu,p}^{\gamma,\lambda} f(z)} + \alpha = \frac{(2p + c - 2\alpha) \omega'(z)}{c - (2p + c - 2\alpha) \omega(z)} + \frac{\omega'(z)}{1 - \omega(z)}
\]  

(46)

so that,

\[
\frac{z \left( J_{\mu,p}^{\gamma,\lambda} f(z) \right)'}{J_{\mu,p}^{\gamma,\lambda} f(z)} + \alpha = (\alpha - p) \frac{1 + \omega(z)}{1 - \omega(z)} - \frac{(2p + c - 2\alpha) \omega'(z)}{c - (2p + c - 2\alpha) \omega(z)} + \frac{\omega'(z)}{1 - \omega(z)}
\]  

(47)
Now, assuming that \( \max_{|z| \leq |z_0|} |\omega(z)| = |\omega(z_0)| = 1 \), \((z \in U^*) \) and applying Jack’s lemma, we obtain
\[
z_0 \omega'(z_0) = \zeta \omega(z_0)
\]
If we set \( \omega(z_0) = e^{i\theta}, (\cos \theta < 0) \) in (47) and observe that
\[
\text{Re} \left\{ (\alpha - p) \frac{1 + \omega(z)}{1 - \omega(z)} \right\} = 0
\]
then, we obtain
\[
\text{Re} \left\{ \frac{z \left( J_{\mu,\lambda}^\gamma f(z) \right)'}{J_{\mu,\lambda}^\gamma f(z)} + \alpha \right\} = \text{Re} \left\{ \frac{z_0 \omega'(z_0)}{1 - \omega(z_0)} - \frac{(2p + c - 2\alpha)z_0 \omega'(z_0)}{c - (2p + c - 2\alpha)\omega(z_0)} \right\}
\]
\[
= \text{Re} \left\{ \frac{\zeta e^{i\theta}}{1 - e^{i\theta}} - \frac{(2p + c - 2\alpha)\zeta e^{i\theta}}{c - (2p + c - 2\alpha)e^{i\theta}} \right\}
\]
\[
= \frac{2\zeta(c + p - \alpha)(p - \alpha)}{c^2 + (2p + c - 2\alpha)^2 - 2c(2p + c - 2\alpha)\cos \theta}
\]
which obviously contradicts the hypothesis \( f(z) \in \Sigma S_{\mu,\lambda}^{\gamma,\lambda}(\alpha) \). Consequently, we can deduce that \(|\omega(z)| < 1 \) \((z \in U^*) \), which, in view of (44), proves the integral-preserving property asserted by Theorem 5.

**Theorem 8.** If \( f(z) \in \Sigma C_{\mu,\lambda}^{\gamma,\lambda}(\alpha) \), Then
\[
\mathcal{L}_{c,p} f(z) \in \Sigma C_{\mu,\lambda}^{\gamma,\lambda}(\alpha).
\]

**Proof.** Suppose that \( f(z) \in \Sigma C_{\mu,\lambda}^{\gamma,\lambda}(\alpha) \), then
\[
z f'(z) \in \Sigma S_{\mu,\lambda}^{\gamma,\lambda}(\alpha)
\]
by applying Theorem 5, we have
\[
\mathcal{L}_{c,p} \left( z f'(z) \right) \in \Sigma S_{\mu,\lambda}^{\gamma,\lambda}(\alpha)
\]
and so,
\[
z \left( \mathcal{L}_{c,p} f(z) \right)' \in \Sigma S_{\mu,\lambda}^{\gamma,\lambda}(\alpha)
\]
which is equivalent to,
\[
\mathcal{L}_{c,p} f(z) \in \Sigma C_{\mu,\lambda}^{\gamma,\lambda}(\alpha)
\]
The proof is completed.
Theorem 9. If \( f(z) \in \Sigma K_{\mu,p}^\gamma(\beta,\alpha) \), Then
\[
\mathcal{L}_{c,p} f(z) \in \Sigma K_{\mu,p}^\gamma(\beta,\alpha).
\]

Proof. Suppose that \( f(z) \in \Sigma K_{\mu,p}^\gamma(\beta,\alpha) \). Then, there exist a function \( g(z) \in \Sigma S_{\mu,p}^\gamma(\alpha) \) such that
\[
\text{Re} \left\{ \frac{z \left( J_{\mu,p}^\gamma \mathcal{L}_{c,p} f(z) \right)'}{J_{\mu,p}^\gamma \mathcal{L}_{c,p} g(z)} \right\} < -\alpha
\]
Let us now setting,
\[
z \left( J_{\mu,p}^\gamma \mathcal{L}_{c,p} f(z) \right)' + \beta = -(p - \beta)q(z) \tag{48}
\]
where \( q(z) \) is given by (22), we find from (41) that
\[
\frac{z \left( J_{\mu,p}^\gamma f(z) \right)'}{J_{\mu,p}^\gamma g(z)} = \frac{J_{\mu,p}^\gamma (zf'(z))}{J_{\mu,p}^\gamma g(z)} \tag{49}
\]
\[
= \frac{z \left( J_{\mu,p}^\gamma \mathcal{L}_{c,p} (zf'(z)) \right)'}{J_{\mu,p}^\gamma \mathcal{L}_{c,p} g(z)} + (p + c) \frac{J_{\mu,p}^\gamma \mathcal{L}_{c,p} (zf'(z))}{J_{\mu,p}^\gamma \mathcal{L}_{c,p} g(z)}
\]
\[
= \frac{z \left( J_{\mu,p}^\gamma \mathcal{L}_{c,p} g(z) \right)'}{J_{\mu,p}^\gamma \mathcal{L}_{c,p} g(z)} + (p + c)
\]
Since \( g(z) \in \Sigma S_{\mu,p}^\gamma(\alpha) \), then according to Theorem 5 we have \( \mathcal{L}_{c,p} g(z) \in \Sigma S_{\mu,p}^\gamma(\alpha) \). Then, we can set
\[
\frac{z \left( J_{\mu,p}^\gamma \mathcal{L}_{c,p} g(z) \right)'}{J_{\mu,p}^\gamma \mathcal{L}_{c,p} g(z)} + \alpha = -(p - \alpha)Q(z) \tag{50}
\]
where \( \text{Re} \{Q(z)\} > 0 \). Equation (48) can be written as,
\[
J_{\mu,p}^\gamma \mathcal{L}_{c,p} (zf'(z)) = (-\beta - (p - \beta)q(z)) J_{\mu,p}^\gamma \mathcal{L}_{c,p} g(z) \tag{51}
\]
By differentiating both sides of (48) with respect to \( z \), we get
\[
\frac{z \left( J_{\mu,p}^\gamma \mathcal{L}_{c,p} (zf'(z)) \right)'}{\left( J_{\mu,p}^\gamma \mathcal{L}_{c,p} g(z) \right)} = -zq'(z)(p - \beta) + (-\beta - (p - \beta)q(z)) \frac{z \left( J_{\mu,p}^\gamma \mathcal{L}_{c,p} g(z) \right)'}{\left( J_{\mu,p}^\gamma \mathcal{L}_{c,p} g(z) \right)} \tag{52}
\]
\[
= -zq'(z)(p - \beta) + (\beta + (p - \beta)q(z))(\alpha + (p - \alpha)Q(z))
\]
Then, by substituting (48), (50) and (52) into (49), we have
\[
z \frac{\left( J_{\mu,p}^{\gamma,\lambda} f(z) \right)'}{J_{\mu,p}^{\gamma,\lambda} g(z)} + \beta = -\left( p - \beta \right)q(z) - \frac{zq'(z)(p - \beta)}{(p + c - \alpha) - (p - \alpha)Q(z)}
\]
(53)

Then, by setting \( u = q(z) = u_1 + iu_2 \) and \( v = zq'(z) = v_1 + iv_2 \), we can define the function \( \Omega(u, v) \) by
\[
\Omega(u, v) = -(p - \beta)u - \frac{(p - \beta)v}{(p + c - \alpha) - (p - \alpha)Q(z)}
\]
(54)

where \((u, v) \in D \subset \mathbb{C} \times \mathbb{C}\). The remainder of our proof of Theorem 7 is similar to that of Theorem 3, so we choose to omit the analogous details involved.

**Theorem 10.** If \( f(z) \in \Sigma K_{\mu,p}^{\gamma,\lambda}(\beta, \alpha) \), then
\[
\mathcal{L}_{\nu,p}f(z) \in \Sigma K_{\mu,p}^{\gamma,\lambda}(\beta, \alpha).
\]

**Proof.** Just as we derived Theorem 6 from Theorem 5. Easily, we can deduce Theorem 8 from Theorem 7. So we choose to omit the proof.

**References**


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