NEW UNIVALENCE CRITERIA FOR AN INTEGRAL OPERATOR
WITH MOCANU’S AND ŞERB’S LEMMA

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ABSTRACT. In this paper we consider an integral operator for analytic functions
in the open unit disk $U$ and we obtain sufficient conditions for univalence of this
integral operator, using Mocanu’s and Şerb’s Lemma.

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1. Introduction

Let $A$ be the class of the functions $f$ which are analytic in the open unit disk
$U = \{z \in \mathbb{C} : |z| < 1\}$ and $f(0) = f'(0) - 1 = 0$.

We denote by $S$ the subclass of $A$ consisting of functions $f \in A$, which are
univalent in $U$.

We consider the integral operator

$$
T_n(z) = \delta \int_0^z t^{\delta-1} \prod_{i=1}^n \left[ \left( \frac{f_i(t)}{t} \right)^{\alpha_i-1} \cdot \left( g_i'(t) \right)^{\beta_i} \cdot \left( \frac{h_i(t)}{k_i(t)} \right)^{\gamma_i} \cdot \left( \frac{h_i'(t)}{k_i'(t)} \right)^{\delta_i} \right] \, dt
$$

for $f_i, g_i, h_i, k_i \in A$ and the complex numbers $\delta, \alpha_i, \beta_i, \gamma_i, \delta_i$, with $\delta \neq 0$, $i = 1, n$, $n \in \mathbb{N} \setminus \{0\}$.
2. Preliminary results

In order to prove main results we will use the following lemmas.

**Lemma 1.** [7] Let $\gamma, \delta$ be complex numbers, $\text{Re} \gamma > 0$ and $f \in A$. If

$$\frac{1 - |z|^{2 \text{Re} \gamma}}{\text{Re} \gamma} \left| \frac{zf''(z)}{f'(z)} \right| \leq 1,$$

for all $z \in U$, then for any complex number $\delta$, $\text{Re} \delta \geq \text{Re} \gamma$, the function $F_\delta$ defined by

$$F_\delta(z) = \left( \delta \int_0^z t^{\delta - 1} f'(t) dt \right)^{\frac{1}{\delta}},$$

is regular and univalent in $U$.

**Lemma 2.** [5] Let $M_0 = 1, 5936...$ the positive solution of equation

$$(2 - M) e^M = 2. \quad (2)$$

If $f \in A$ and

$$\left| \frac{f''(z)}{f'(z)} \right| \leq M_0,$$

for $z \in U$, then

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| \leq 1, \quad (z \in U)$$

The edge $M_0$ is sharp.

**Lemma 3.** [3] Let $f$ be the function regular in the disk $U_R = \{ z \in \mathbb{C} : |z| < R \}$ with $|f(z)| < M$, $M$ fixed. If $f(z)$ has in $z = 0$ one zero with multiply $\geq m$, then

$$|f(z)| \leq \frac{M}{R^m} z^m,$$

the equality for $z \neq 0$ can hold only if

$$f(z) = e^{i\theta} \frac{M}{R^m} z^m,$$

where $\theta$ is constant.
3. Main Results

**Theorem 4.** Let $\gamma, \delta, \alpha_i, \beta_i, \gamma_i, \delta_i$ be complex numbers, $c = \Re \gamma > 0$, $M_0$ the positive solution of the equation (2), $M_0 = 1,5936...$ and $f_i, g_i, h_i, k_i \in A$, $f_i(z) = z + a_{2i}z^2 + a_{3i}z^3 + ..., \quad g_i(z) = z + b_{2i}z^2 + b_{3i}z^3 + ..., \quad h_i(z) = z + c_{2i}z^2 + c_{3i}z^3 + ..., \quad k_i(z) = z + d_{2i}z^2 + d_{3i}z^3 + ..., \quad i = \overline{1, n}$ If

$$\left| \frac{f''_i(z)}{f'_i(z)} \right| \leq M_0, \quad \left| \frac{g''_i(z)}{g'_i(z)} \right| \leq M_0, \quad \left| \frac{h''_i(z)}{h'_i(z)} \right| \leq M_0, \quad \left| \frac{k''_i(z)}{k'_i(z)} \right| \leq M_0, \tag{3}$$

for all $z \in U$, $i = \overline{1, n}$ and

$$\frac{1}{c} \sum_{i=1}^{n} |\alpha_i - 1| + \frac{2M_0}{2c+1} \sum_{i=1}^{n} |\beta_i| + \frac{2}{c} \sum_{i=1}^{n} |\gamma_i| + \frac{4M_0}{2c+1} \sum_{i=1}^{n} |\delta_i| \leq 1, \tag{4}$$

then for all $\delta$ complex numbers, $Re\delta \geq Re\gamma$, the integral operator $T_n$, given by (1) is in the class $S$.

**Proof.** Let us define the function

$$H_n(z) = \int_{0}^{z} \prod_{i=1}^{n} \left[ \left( \frac{f_i(t)}{t} \right)^{\alpha_i-1} \left( \frac{g_i(t)}{t} \right)^{\beta_i} \left( \frac{h_i(t)}{k_i(t)} \right)^{\gamma_i} \left( \frac{h'_i(t)}{k'_i(t)} \right)^{\delta_i} \right] dt,$$

for $f_i, g_i, h_i, k_i \in A$, $i = \overline{1, n}$.

The function $H_n$ is regular in $U$ and satisfy the following usual normalization conditions $H_n(0) = H'_n(0) - 1 = 0$.

We have

$$\frac{zH''_n(z)}{H'_n(z)} = \sum_{i=1}^{n} \left[ (\alpha_i - 1) \left( \frac{zf'_i(z)}{f_i(z)} - 1 \right) + \beta_i \frac{zg''_i(z)}{g'_i(z)} \right] +$$

$$+ \sum_{i=1}^{n} \left[ \gamma_i \left( \frac{zh'_i(z)}{h_i(z)} - \frac{zk'_i(z)}{k_i(z)} \right) + \delta_i \left( \frac{zh''_i(z)}{h'_i(z)} - \frac{zk''_i(z)}{k'_i(z)} \right) \right],$$

for all $z \in U$.

Therefore

$$\frac{1}{c} \left| \frac{zH''_n(z)}{H'_n(z)} \right| \leq \frac{1}{c} \left| z \right|^{2c} \sum_{i=1}^{n} \left[ |\alpha_i - 1| \left| \frac{zf'_i(z)}{f_i(z)} - 1 \right| + |\beta_i| \left| \frac{zg''_i(z)}{g'_i(z)} \right| \right] +$$

$$+ \sum_{i=1}^{n} \left| \gamma_i \left( \frac{zh'_i(z)}{h_i(z)} - \frac{zk'_i(z)}{k_i(z)} \right) + \delta_i \left( \frac{zh''_i(z)}{h'_i(z)} - \frac{zk''_i(z)}{k'_i(z)} \right) \right|,$$
+ \left[ |\gamma_i| \left( \left| \frac{zh''_i(z)}{h_i(z)} - 1 \right| + \left| \frac{zk''_i(z)}{k_i(z)} - 1 \right| \right) + |\delta_i| \left( \left| \frac{zh''_i(z)}{h'_i(z)} \right| + \left| \frac{zk''_i(z)}{k'_i(z)} \right| \right) \right], \quad (5)

for all \( z \in \mathcal{U} \).

Using (3), (4) and Lemma Mocanu and Şerb, from (5) we get
\[
\left| \frac{zf'_i(z)}{f_i(z)} - 1 \right| < 1, \quad \left| \frac{zh'_i(z)}{h_i(z)} - 1 \right| < 1, \quad \left| \frac{zk'_i(z)}{k_i(z)} - 1 \right| < 1,
\]
for all \( z \in \mathcal{U}, i = 1, n \) and hence, we have
\[
1 - |z|^{2c} c \left| \frac{zH''_n(z)}{H'_n(z)} \right| \leq 1 - |z|^{2c} c \sum_{i=1}^{n} |\alpha_i - 1| + \\
1 - |z|^{2c} c |z| M_0 \sum_{i=1}^{n} |\beta_i| + 1 - |z|^{2c} c 2 \sum_{i=1}^{n} |\gamma_i| + 1 - |z|^{2c} c |z| 2M_0 \sum_{i=1}^{n} |\delta_i|, \quad (6)
\]
for all \( z \in \mathcal{U} \).

Since
\[
\max_{|z| \leq 1} \left( 1 - |z|^{2c} \right) \frac{|z|}{c} = \frac{2}{(2c + 1)^{\frac{2c+1}{2c}}}, \quad (7)
\]
from (6) and (7) we obtain
\[
1 - |z|^{2c} c \left| \frac{zH''_n(z)}{H'_n(z)} \right| \leq \\
\frac{1}{c} \sum_{i=1}^{n} |\alpha_i - 1| + \frac{2M_0}{(2c + 1)^{\frac{2c+1}{2c}}} \sum_{i=1}^{n} |\beta_i| + \frac{2}{c} \sum_{i=1}^{n} |\gamma_i| + \frac{4M_0}{(2c + 1)^{\frac{2c+1}{2c}}} \sum_{i=1}^{n} |\delta_i|, \quad (8)
\]
for all \( z \in \mathcal{U}, i = 1, n \).

Using (6), from (8) we have
\[
1 - |z|^{2c} c \left| \frac{zH''_n(z)}{H'_n(z)} \right| \leq 1. \quad (9)
\]

Now, from (9), by Lemma 2.1, it results that the integral operator \( \mathcal{T}_n \), given by (1) is in the class \( \mathcal{S} \).
Letting $\delta = 1$ in Theorem 3.1, we have

**Corollary 5.** Let $\gamma, \alpha_1, \beta_i, \gamma_i, \delta_i$ be complex numbers, $0 < \Re \gamma \leq 1$, $c = \Re \gamma$, $M_0$ the positive solution of the equation (3), $M_0 = 1,5936...$ and $f_i, g_i, h_i, k_i \in \mathcal{A}$, $f_i(z) = z + a_2 z^2 + a_3 z^3 + ..., g_i(z) = z + b_2 z^2 + b_3 z^3 + ..., h_i(z) = z + c_2 z^2 + c_3 z^3 + ..., k_i(z) = z + d_2 z^2 + d_3 z^3 + ..., i = 1, n$.

If

$$\left| \frac{f_i''(z)}{f_i'(z)} \right| \leq M_0, \quad \left| \frac{g_i''(z)}{g_i'(z)} \right| \leq M_0, \quad \left| \frac{h_i''(z)}{h_i'(z)} \right| \leq M_0, \quad \left| \frac{k_i''(z)}{k_i'(z)} \right| \leq M_0,$$

for all $z \in \mathcal{U}$, $i = 1, n$ and

$$\frac{1}{c} \sum_{i=1}^{n} |\alpha_i - 1| + \frac{2M_0}{(2c + 1)^{2+1}} \sum_{i=1}^{n} |\beta_i| + \frac{2}{c} \sum_{i=1}^{n} |\gamma_i| + \frac{4M_0}{(2c + 1)^{2+1}} \sum_{i=1}^{n} |\delta_i| \leq 1,$$

then the integral operator $F_n$ defined by

$$F_n(z) = \int_{0}^{z} \prod_{i=1}^{n} \left[ \left( \frac{f_i(t)}{t} \right)^{\alpha_i - 1} \cdot \left( \frac{g_i(t)}{t} \right)^{\beta_i} \cdot \left( \frac{h_i(t)}{t} \right)^{\gamma_i} \cdot \left( \frac{k_i(t)}{t} \right)^{\delta_i} \right] dt,$$  \hspace{1cm} (10)

is in the class $\mathcal{S}$.

Letting $\delta = 1$ and $\delta_1 = \delta_2 = ... = \delta_n = 0$ in Theorem 3.1, we have

**Corollary 6.** Let $\gamma, \alpha_i, \beta_i, \gamma_i$ be complex numbers, $0 < \Re \gamma \leq 1$, $c = \Re \gamma$, $M_0$ the positive solution of the equation (2), $M_0 = 1,5936...$ and $f_i, g_i, h_i, k_i \in \mathcal{A}$, $f_i(z) = z + a_2 z^2 + a_3 z^3 + ..., g_i(z) = z + b_2 z^2 + b_3 z^3 + ..., h_i(z) = z + c_2 z^2 + c_3 z^3 + ..., k_i(z) = z + d_2 z^2 + d_3 z^3 + ..., i = 1, n$.

If

$$\left| \frac{f_i''(z)}{f_i'(z)} \right| \leq M_0, \quad \left| \frac{g_i''(z)}{g_i'(z)} \right| \leq M_0, \quad \left| \frac{h_i''(z)}{h_i'(z)} \right| \leq M_0, \quad \left| \frac{k_i''(z)}{k_i'(z)} \right| \leq M_0,$$

for all $z \in \mathcal{U}$, $i = 1, n$ and

$$\frac{1}{c} \sum_{i=1}^{n} |\alpha_i - 1| + \frac{2M_0}{(2c + 1)^{2+1}} \sum_{i=1}^{n} |\beta_i| + \frac{2}{c} \sum_{i=1}^{n} |\gamma_i| \leq 1,$$

then the integral operator $S_n$ defined by

$$S_n(z) = \int_{0}^{z} \prod_{i=1}^{n} \left[ \left( \frac{f_i(t)}{t} \right)^{\alpha_i - 1} \cdot \left( \frac{g_i(t)}{t} \right)^{\beta_i} \cdot \left( \frac{h_i(t)}{t} \right)^{\gamma_i} \cdot \left( \frac{k_i(t)}{t} \right)^{\delta_i} \right] dt,$$  \hspace{1cm} (11)

is in the class $\mathcal{S}$.
Letting $\delta = 1$ and $\beta_1 = \beta_2 = \ldots = \beta_n = 0$ in Theorem 3.1, we obtain

**Corollary 7.** Let $\gamma, \alpha_i, \gamma_i, \delta_i$ be complex numbers, $0 < \text{Re}\gamma \leq 1$, $c = \text{Re}\gamma$, $i = \overline{1,n}$, $M_0$ the positive solution of the equation (2), $M_0 = 1,5936\ldots$ and $f_i, h_i, k_i \in A$, $f_i(z) = z + a_{2i}z^2 + a_{3i}z^3 + \ldots$, $h_i(z) = z + c_{2i}z^2 + c_{3i}z^3 + \ldots$, $k_i(z) = z + d_{2i}z^2 + d_{3i}z^3 + \ldots$, $i = \overline{1,n}$.

If

$$\left| \frac{f_i''(z)}{f_i'(z)} \right| \leq M_0, \quad \left| \frac{h_i''(z)}{h_i'(z)} \right| \leq M_0, \quad \left| \frac{k_i''(z)}{k_i'(z)} \right| \leq M_0,$$

for all $z \in U$, $i = \overline{1,n}$ and

$$\frac{1}{c} \sum_{i=1}^{n} |\alpha_i - 1| + \frac{2}{c} \sum_{i=1}^{n} |\gamma_i| + \frac{4M_0}{(2c + 1)^{2c}} \sum_{i=1}^{n} |\delta_i| \leq 1,$$

then the integral operator $X_n$ defined by

$$X_n(z) = \int_{0}^{z} \prod_{i=1}^{n} \left( \frac{f_i(t)}{t} \right)^{\alpha_i - 1} \cdot \left( \frac{h_i(t)}{k_i(t)} \right)^{\gamma_i} \cdot \left( \frac{h_i'(t)}{k_i'(t)} \right)^{\delta_i} \, dt,$$  \hspace{1cm} (12)

is in the class $S$.

Letting $\delta = 1$ and $\alpha_1 = \alpha_2 = \ldots = \alpha_n = 0$ in Theorem 3.1, we have

**Corollary 8.** Let $\gamma, \beta_i, \gamma_i, \delta_i$ be complex numbers, $0 < \text{Re}\gamma \leq 1$, $c = \text{Re}\gamma$, $M_0$ the positive solution of the equation (2), $M_0 = 1,5936\ldots$ and $g_i, h_i, k_i \in A$, $g_i(z) = z + b_{2i}z^2 + b_{3i}z^3 + \ldots$, $h_i(z) = z + c_{2i}z^2 + c_{3i}z^3 + \ldots$, $k_i(z) = z + d_{2i}z^2 + d_{3i}z^3 + \ldots$, $i = \overline{1,n}$.

If

$$\left| \frac{g_i''(z)}{g_i'(z)} \right| \leq M_0, \quad \left| \frac{h_i''(z)}{h_i'(z)} \right| \leq M_0, \quad \left| \frac{k_i''(z)}{k_i'(z)} \right| \leq M_0,$$

for all $z \in U$, $i = \overline{1,n}$ and

$$\frac{2M_0}{(2c + 1)^{2c}} \sum_{i=1}^{n} |\beta_i| + \frac{2}{c} \sum_{i=1}^{n} |\gamma_i| + \frac{4M_0}{(2c + 1)^{2c}} \sum_{i=1}^{n} |\delta_i| \leq 1,$$

then the integral operator $D_n$ defined by

$$D_n(z) = \int_{0}^{z} \prod_{i=1}^{n} \left( \frac{g_i(t)}{t} \right)^{\beta_i} \cdot \left( \frac{h_i(t)}{k_i(t)} \right)^{\gamma_i} \cdot \left( \frac{h_i'(t)}{k_i'(t)} \right)^{\delta_i} \, dt,$$  \hspace{1cm} (13)

is in the class $S$. 

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Letting $\delta = 1$ and $\gamma_1 = \gamma_2 = \ldots = \gamma_n = 0$ in Theorem 3.1, we have

**Corollary 9.** Let $\gamma, \alpha_i, \beta_i, \delta_i$ be complex numbers, $0 < Re\gamma \leq 1$, $c = Re\gamma$, $M_0$ the positive solution of the equation (2), $M_0 = 1.5936...$ and $f_i, g_i, h_i, k_i \in \mathcal{A}$, $f_i(z) = z + a_{2i}z^2 + a_{3i}z^3 + \ldots$, $g_i(z) = z + b_{2i}z^2 + b_{3i}z^3 + \ldots$, $h_i(z) = z + c_{2i}z^2 + c_{3i}z^3 + \ldots$, $k_i(z) = z + d_{2i}z^2 + d_{3i}z^3 + \ldots$, $i = 1, n$.

If
\[
\frac{|f''(z)|}{f'(z)} \leq M_0, \quad \frac{|g''(z)|}{g'(z)} \leq M_0, \quad \frac{|h''(z)|}{h'(z)} \leq M_0, \quad \frac{|k''(z)|}{k'(z)} \leq M_0,
\]
for all $z \in \mathcal{U}$, $i = 1, n$ and
\[
\frac{1}{e} \sum_{i=1}^{n} |\alpha_i - 1| + \frac{2M_0}{(2c + 1)^{2c+1}} \sum_{i=1}^{n} |\beta_i| + \frac{4M_0}{(2c + 1)^{2c+1}} \sum_{i=1}^{n} |\delta_i| \leq 1,
\]
then the integral operator $Y_n$ defined by
\[
Y_n(z) = \int_0^z \prod_{i=1}^{n} \left[ \left( \frac{f_i(t)}{t} \right)^{\alpha_i-1} \cdot (g_i'(t))^{\beta_i} \cdot \left( \frac{h_i(t)}{k_i(t)} \right)^{\delta_i} \right] dt, \quad (14)
\]
is in the class $\mathcal{S}$.

Letting $n = 1$, $\delta = \gamma = \alpha$ and $\alpha_i - 1 = \beta_i = \gamma_i$ in Theorem 3.1, we obtain

**Corollary 10.** Let $\alpha$ be complex number, $a = Re\alpha > 0$, $M_0$ the positive solution of the equation (2), $M_0 = 1.5936...$ and $f, g, h, k \in \mathcal{A}$, $f(z) = z + a_2z^2 + a_3z^3 + \ldots$, $g(z) = z + b_2z^2 + b_3z^3 + \ldots$, $h(z) = z + c_2z^2 + c_3z^3 + \ldots$, $k(z) = z + d_2z^2 + d_3z^3 + \ldots$.

If
\[
\frac{|f''(z)|}{f'(z)} \leq M_0, \quad \frac{|g''(z)|}{g'(z)} \leq M_0, \quad \frac{|h''(z)|}{h'(z)} \leq M_0, \quad \frac{|k''(z)|}{k'(z)} \leq M_0,
\]
for all $z \in \mathcal{U}$, and
\[
\frac{\alpha - 1}{a} + \frac{2\beta M_0 a^{2a+1}}{(2a + 1)^{2a+1}} + \frac{2\gamma}{a} + \frac{4\delta M_0 a^{2a+1}}{(2a + 1)^{2a+1}} \leq 1,
\]
then the integral operator $T$ defined by
\[
T(z) = \left[ \alpha \int_0^z t^{\alpha-1} \left( f(t) \cdot g'(t) \cdot \frac{h(t)}{k(t)} \cdot \frac{h'(t)}{k'(t)} \right)^{\alpha-1} dt \right]^\frac{1}{\alpha}, \quad (15)
\]
is in the class $\mathcal{S}$.
Letting $M_0 = M$ from (3) in Theorem 3.1, we obtain

**Corollary 11.** Let $\gamma, \delta, \alpha_i, \beta_i, \gamma_i, \delta_i$ be complex numbers, $c = \text{Re}\gamma > 0$, $M$ a positive number and $M_0$ the positive solution of the equation (2), $M_0 = 1, 5936...$ and $f_i, g_i, h_i, k_i \in \mathcal{A}$, $f_i(z) = z + a_{2i}z^2 + a_{3i}z^3 + \ldots$, $g_i(z) = z + b_{2i}z^2 + b_{3i}z^3 + \ldots$, $h_i(z) = z + c_{2i}z^2 + c_{3i}z^3 + \ldots$, $k_i(z) = z + d_{2i}z^2 + d_{3i}z^3 + \ldots$, $i = 1, n$

If

$$
\left| \frac{f''_i(z)}{f'_i(z)} \right| \leq M_0, \quad \left| \frac{g''_i(z)}{g'_i(z)} \right| \leq M_0, \quad \left| \frac{h''_i(z)}{h'_i(z)} \right| \leq M_0, \quad \left| \frac{k''_i(z)}{k'_i(z)} \right| \leq M_0,
$$

for all $z \in \mathcal{U}, i = 1, n$ and

$$
\frac{1}{c} \sum_{i=1}^{n} |\alpha_i - 1| + \frac{2M}{c} \sum_{i=1}^{n} |\beta_i| + \frac{2}{c} \sum_{i=1}^{n} |\gamma_i| + \frac{4M_0}{(2c+1)^{2c+2}} \sum_{i=1}^{n} |\delta_i| \leq 1,
$$

then $f_i, g_i, h_i, k_i \in \mathcal{S}$, $i = 1, n$ and for all $\delta$ complex numbers, $\text{Re}\delta \geq \text{Re}\gamma$, the integral operator $\mathcal{T}_n$, given by (1) is in the class $\mathcal{S}$.

**References**


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