GENERALIZED VISCOSITY APPROXIMATION METHOD FOR EQUILIBRIUM AND FIXED POINT PROBLEMS

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Abstract. In this paper, we introduce a new iterative scheme by the generalized viscosity approximation method for finding a common element of the set of solutions of an equilibrium problem and the set of common fixed points of infinitely many nonexpansive mappings in a Hilbert space. Then, we prove a strong convergence theorem which improves and extends some recent results.

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1. Introduction

Let $H$ be a real Hilbert space, let $A$ be a bounded operator on $H$. In this paper, we assume that $A$ is strongly positive; that is, there exists a constant $\gamma > 0$ such that $\langle Ax, x \rangle \geq \gamma \|x\|^2$, $\forall x \in H$. Let $C$ be a nonempty closed convex subset of $H$ and $\phi : C \times C \to \mathbb{R}$ be a bifunction of $C \times C$ into $\mathbb{R}$. The equilibrium problem for $\phi : C \times C \to \mathbb{R}$ is to find $u \in C$ such that

$$\phi(u, v) \geq 0 \quad \text{for all} \quad v \in C. \quad (1)$$

The set of solutions of (1) is denoted by $EP(\phi)$. The equilibrium problem (1) includes as special cases numerous problems in physics, optimization and economics. Some authors have proposed some useful methods for solving the equilibrium problem (1); see [6], [10] and [18].

A mapping $T$ of $H$ into itself is called nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in H$. Let $F(T)$ denote the fixed points set of $T$. Also, a contraction on $H$ is a self-mapping $f$ of $H$ such that $\|f(x) - f(y)\| \leq \alpha \|x - y\|$ for all $x, y \in H$, where $\alpha \in [0, 1)$ is a constant. In 2000, Moudafi [15] proved the following strong convergence theorem.
Theorem 1. [15] Let $C$ be a nonempty closed convex subset of a Hilbert space $H$ and let $T$ be a nonexpansive self-mapping on $C$ such that $F(T) \neq \emptyset$. Let $f : C \to C$ be a contraction and let $\{x_n\}$ be a sequence defined as follows: $x_1 = x \in C$ and

$$x_{n+1} = \frac{1}{1+\varepsilon_n}Tx_n + \frac{\varepsilon_n}{1+\varepsilon_n}f(x_n)$$

for all $n \geq 1$, where $\varepsilon_n \subset (0,1)$ satisfies

$$\lim_{n \to \infty} \varepsilon_n = 0, \sum_{n=1}^{\infty} \varepsilon_n = \infty \text{ and } \lim_{n \to \infty} \frac{1}{\varepsilon_{n+1}} - \frac{1}{\varepsilon_n} = 0.$$

Then, the sequence $\{x_n\}$ converges strongly to $z \in F(T)$, where $z = P_{F(T)}f(z)$ and $P_{F(T)}$ is the metric projection of $H$ onto $F$.

Such a method for approximation of fixed points is called the viscosity approximation method.

Finding an optimal point in the intersection $F$ of the fixed points set of a family of nonexpansive mappings is one that occurs frequently in various areas of mathematical sciences and engineering. For example, the well-known convex feasibility problem reduces to finding a point in the intersection of the fixed points set of a family of nonexpansive mappings; see, e.g., [2] and [5]. The problem of finding an optimal point that minimizes a given cost function $\Theta : H \to \mathbb{R}$ over $F$ is of wide interdisciplinary interest and practical importance see, e.g., [1], [4], [8] and [24].

A simple algorithmic solution to the problem of minimizing a quadratic function over $F$ is of extreme value in many applications including the set theoretic signal estimation, see, e.g., [11] and [24]. The best approximation problem of finding the projection $P_{F}(a)$ (in the norm induced by inner product of $H$) from any given point $a$ in $H$ is the simplest case of our problem.

Yao et al. [22] introduced the iterative sequence:

$$x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + ((1-\beta_n)I-\alpha_n A)W_n x_n \text{ for all } n \geq 0.$$ 

where $f$ is a contraction on $H$, $A : H \to H$ is a strongly positive bounded linear operator, $\gamma > 0$ is a constant, $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in $(0,1)$, $W_n$ is the $W$-mapping generated by an infinite countable family of nonexpansive mappings $T_1, T_2, \ldots, T_n, \ldots$ and $\lambda_1, \lambda_2, \ldots, \lambda_n, \ldots$ such that the common fixed points set $F := \bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$. Under very mild conditions on the parameters, it was proved that the sequence $\{x_n\}$ converges strongly to $p \in F$ where $p$ is the unique solution in $F$ of the following variational inequality:

$$\langle (A-\gamma f)p, p - x^* \rangle \leq 0 \text{ for all } x^* \in F,$$
which is the optimality condition for minimization problem
\[
\min_{x \in F} \frac{1}{2} \langle Ax, x \rangle - h(x),
\]
where \( h \) is a potential function for \( \gamma f \) (i.e., \( h'(x) = \gamma f(x) \) for \( x \in H \)).

On the other hand, Ceng and Yao \([7]\) introduced an iterative scheme by
\[
\begin{align*}
\phi(u_n, x) + \frac{1}{r_n} \langle x - u_n, u_n - x_n \rangle & \geq 0, \quad \text{for all } x \in C, \\
y_n &= (1 - \gamma_n) x_n + \gamma_n W_n u_n, \\
x_{n+1} &= \beta_n W_n y_n + \alpha_n f(y_n) + (1 - \beta_n - \alpha_n) x_n,
\end{align*}
\]
where \( \{\alpha_n\}, \{\beta_n\} \) and \( \{\gamma_n\} \) are three sequences in \((0, 1)\) such that \( \alpha_n + \beta_n \leq 1 \) and \( W_n \) is the \( W \)-mapping generated by an infinite countable family of nonexpansive mappings \( T_1, T_2, \ldots, T_n, \ldots \) and \( \lambda_1, \lambda_2, \ldots, \lambda_n, \ldots \).

Razani and Yazdi \([16]\), motivated by Yao et al. \([22]\) and Ceng and Yao \([7]\), introduced a new iterative scheme by the viscosity approximation method:
\[
\begin{align*}
\phi(u_n, x) + \frac{1}{r_n} \langle x - u_n, u_n - x_n \rangle & \geq 0, \quad \text{for all } x \in C, \\
y_n &= (1 - \gamma_n) x_n + \gamma_n W_n u_n, \\
x_{n+1} &= \alpha_n \gamma f(y_n) + \beta_n x_n + ((1 - \beta_n) I - \alpha_n A) W_n y_n,
\end{align*}
\]
where \( \{\alpha_n\}, \{\beta_n\} \) and \( \{\gamma_n\} \) are three sequences in \((0, 1)\), \( f \) is a contraction, \( A \) is a strongly positive bounded linear operator, \( \gamma > 0 \) is a constant and \( W_n \) is the \( W \)-mapping generated by an infinite countable family of nonexpansive mappings \( T_1, T_2, \ldots, T_n, \ldots \) and \( \lambda_1, \lambda_2, \ldots, \lambda_n, \ldots \) such that the common fixed points set \( F := \bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset \). They proved the sequences \( \{x_n\} \) and \( \{u_n\} \) generated iteratively by (3) converge strongly to \( p \in F \), where \( p = P_{\bigcap_{n=1}^{\infty} F(T_n) \cap EP(\phi)}(I - A + \gamma f)(p) \).

Moreover, Duan and He \([9]\) combined a sequence of contractive mappings \( \{f_n\} \) and proposed a generalized viscosity approximation method. They considered the following iterative algorithm:
\[
x_{n+1} = \alpha_n f_n(x_n) + (1 - \alpha_n) T x_n,
\]
where \( T \) is a nonexpansive mapping and \( \{\alpha_n\} \) is a sequence in \((0, 1)\). They proved the sequence \( \{x_n\} \) converges strongly to \( p \in F(T) \) which is a unique solution of a variational inequality.

In this paper, inspired by above results, we introduce a new iterative scheme for finding a common element of the set of solutions of the equilibrium problem (1) and the set of common fixed points of infinitely many nonexpansive mappings in a Hilbert space. Then, we prove a strong convergence theorem which improves the main results of \([7]\) and \([16]\).
2. Preliminaries

Let $H$ be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and the norm $\| \cdot \|$. We denote weak convergence and strong convergence by notation $\rightharpoonup$ and $\rightarrow$, respectively. Let $C$ be a nonempty closed convex subset of $H$. Then, for any $x \in H$, there exists a unique nearest point in $C$, denoted by $P_C(x)$, such that

$$\|x - P_C(x)\| \leq \|x - y\| \text{ for all } y \in C.$$ 

Such a $P_C$ is called the metric projection of $H$ onto $C$. It is known that $P_C$ is nonexpansive. Further, for $x \in H$ and $z \in C$,

$$z = P_C(x) \iff \langle x - z, z - y \rangle \geq 0 \text{ for all } y \in C.$$ 

Now, we collect some lemmas which will be used in the proofs for the main results.

**Lemma 2.** [3] Let $C$ be a nonempty closed convex subset of $H$ and $\phi : C \times C \to \mathbb{R}$ be a bifunction satisfying $(A_1) - (A_4)$. Let $r > 0$ and $x \in H$. Then, there exists $z \in C$ such that

$$\phi(z, y) + \frac{1}{r} \langle x - z, z - x \rangle \geq 0 \text{ for all } y \in C.$$ 

**Lemma 3.** [6] Assume that $\phi : C \times C \to \mathbb{R}$ satisfies $(A_1) - (A_4)$. For $r > 0$ and $x \in H$, define a mapping $T_r : H \to C$ as follows:

$$T_r x = \{ z \in C : \phi(z, y) + \frac{1}{r} \langle y - z, z - x \rangle \geq 0 \text{ for all } y \in C \}$$ 

for all $x \in H$. Then, the following hold:

(i) $T_r$ is single-valued;

(ii) $T_r$ is firmly nonexpansive, i.e., for any $x, y \in H$

$$\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y \rangle;$$

(iii) $F(T_r) = EP(\phi)$;

(iv) $EP(\phi)$ is closed and convex.

**Lemma 4.** [14] Assume $A$ is a strongly positive bounded linear operator on a Hilbert space $H$ with coefficient $\gamma > 0$ and $0 \leq \rho \leq \frac{1}{\|A\|^{-1}}$. Then, $\|I - \rho A\| \leq 1 - \rho \gamma$.

**Lemma 5.** [20] Let $H$ be a real Hilbert space. Then, for all $x, y \in H$ and $\lambda \in [0, 1]$,

$$\|\lambda x + (1 - \lambda)y\|^2 = \lambda \|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)\|x - y\|.$$
Lemma 6. [19] Let \( \{x_n\} \) and \( \{y_n\} \) be bounded sequences in Banach space \( X \) and let \( \{\beta_n\} \) be a sequence in \([0, 1]\) with \( 0 < \liminf_{n \to \infty} \beta_n \leq \limsup_{n \to \infty} \beta_n < 1 \). Suppose \( x_{n+1} = (1 - \beta_n)y_n + \beta_n x_n \) for all integers \( n \geq 0 \) and \( \limsup_{n \to \infty} (\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0 \). Then, \( \lim_{n \to \infty} \|x_n - y_n\| = 0 \).

Lemma 7. [21] Assume \( \{a_n\} \) is a sequence of nonnegative real numbers such that

\[
a_{n+1} \leq (1 - \gamma_n)a_n + \gamma_nv_n,
\]

where \( \{\gamma_n\} \) is a sequence in \((0, 1)\) and \( \{v_n\} \) is a sequence in \( \mathbb{R} \) such that (i) \( \sum_{n=1}^{\infty} \gamma_n = \infty \);

(ii) \( \limsup_{n \to \infty} v_n \leq 0 \) or \( \sum_{n=1}^{\infty} |\gamma_nv_n| < \infty \).

Then, \( \lim_{n \to \infty} a_n = 0 \).

Lemma 8. [15] Assume \( A \) is a strongly positive bounded linear operator on a Hilbert space \( H \) with coefficient \( \gamma > 0 \) and \( 0 < \rho \leq \|A\|^{-1} \). Then \( \|I - \rho A\| \leq 1 - \rho \gamma \).

Lemma 9. [12] Each Hilbert space \( H \) satisfies Opial’s condition, i. e., for any sequence \( \{x_n\} \subset H \) with \( x_n \rightharpoonup x \), the inequality

\[
\liminf_{n \to \infty} \|x_n - x\| < \liminf_{n \to \infty} \|x_n - y\|
\]

holds for each \( y \in H \) with \( x \neq y \).

Let \( H \) be a real Hilbert space and \( A \) be a strongly positive bounded linear operator on \( H \) with coefficient \( \gamma > 0 \). Let \( f \) be a contraction of \( C \) into itself with constant \( \alpha \in [0, 1) \) and \( 0 < \alpha \gamma < \gamma \) where \( \gamma \) is some constant. Let \( \{T_n\}_{n=1}^{\infty} \) be a sequence of nonexpansive self-mappings on \( H \) and \( \{\lambda_n\}_{n=1}^{\infty} \) be a sequence of nonnegative numbers in \([0, 1]\). For any \( n \geq 1 \), define a mapping \( W_n \) of \( H \) into itself as follows:

\[
U_{n,n+1} = I,
U_{n,n} = \lambda_nT_nU_{n,n+1} + (1 - \lambda_n)I,
\]

\[
U_{n,k} = \lambda_kT_kU_{n,k+1} + (1 - \lambda_k)I,
U_{n,k-1} = \lambda_{k-1}T_{k-1}U_{n,k} + (1 - \lambda_{k-1})I,
\]

\[
U_{n,2} = \lambda_2T_2U_{n,3} + (1 - \lambda_2)I,
W_n = U_{n,1} = \lambda_1T_1U_{n,2} + (1 - \lambda_1)I.
\]

Such a mapping \( W_n \) is called the \( W \)-mapping generated by \( T_n, T_{n-1}, \ldots, T_1 \) and \( \lambda_n, \lambda_{n-1}, \ldots, \lambda_1 \); see [13].
Lemma 10. [17] Let $C$ be a nonempty closed convex subset of a strictly convex Banach space $X$, $\{T_n\}_{n=1}^{\infty}$ be a sequence of nonexpansive self-mappings on $C$ such that $\bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$ and $\{\lambda_n\}_{n=1}^{\infty}$ be a sequence of positive numbers in $[0,b]$ for some $b \in (0,1)$. Then, for every $x \in C$ and $k \geq 1$, the limit $\lim_{n \to \infty} U_{n,k}x$ exists.

Remark 1. [23] It can be known from Lemma 10 that if $D$ is a nonempty bounded subset of $C$, then for $\varepsilon > 0$ there exists $n_0 \geq k$ such that for all $n > n_0$

$$\sup_{x \in D} \|U_{n,k}x - U_kx\| \leq \varepsilon.$$  

Remark 2. [23] Using Lemma 10, one can define mapping $W : C \to C$ as follows:

$$Wx = \lim_{n \to \infty} W_nx = \lim_{n \to \infty} U_{n,1}x,$$

for all $x \in C$. Such a $W$ is called the $W$-mapping generated by $\{T_n\}_{n=1}^{\infty}$ and $\{\lambda_n\}_{n=1}^{\infty}$. Since $W$ is nonexpansive, $W : C \to C$ is also nonexpansive.

If $\{x_n\}$ is a bounded sequence in $C$, then we put $D = \{x_n : n \geq 0\}$. Hence, it is clear from Remark 1 that for an arbitrary $\varepsilon > 0$ there exists $N_0 \geq 1$ such that for all $n > N_0$

$$\|W_n x_n - W x_n\| = \|U_{n,1} x_n - U_1 x_n\| \leq \sup_{x \in D} \|U_{n,1} x - U_1 x\| \leq \varepsilon.$$  

This implies that $\lim_{n \to \infty} \|W_n x_n - W x_n\| = 0$.

Throughout this paper, we always assume that $\{\lambda_n\}_{n=1}^{\infty}$ is a sequence of positive numbers in $[0,b]$ for some $b \in (0,1)$.

Lemma 11. [17] Let $C$ be a nonempty closed convex subset of a strictly convex Banach space $X$, $\{T_n\}_{n=1}^{\infty}$ be a sequence of nonexpansive self-mappings on $C$ such that $\bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$ and $\{\lambda_n\}_{n=1}^{\infty}$ be a sequence of positive numbers in $[0,b]$ for some $b \in (0,1)$. Then, $F(W) = \bigcap_{n=1}^{\infty} F(T_n)$.

3. Main result

In this section, we prove the following strong convergence theorem for finding a common element of the set of solutions of the equilibrium problem (1) and the set of common fixed points of infinitely many nonexpansive mappings in a Hilbert space. Suppose the contractive mapping sequence $\{f_n(x)\}$ is uniformly convergent for any $x \in D$, where $D$ is any bounded subset of $C$.  

124
Theorem 12. Let \( C \) be a nonempty closed convex subset of a real Hilbert space \( H \). Let \( \phi : C \times C \to \mathbb{R} \) be a bifunction satisfying (A1) – (A4), \( A \) be a strongly positive bounded linear operator on \( C \) with coefficient \( \gamma > 0 \) and \( \| A \| \leq 1 \) and \( \{ T_n \}_{n=1}^{\infty} \) be an infinite family of nonexpansive self-mappings on \( C \) which satisfies \( F := \bigcap_{n=1}^{\infty} F(T_n) \cap EP(\phi) \neq \emptyset \). Suppose \( \{ \alpha_n \}, \{ \beta_n \} \) and \( \{ \gamma_n \} \) are sequences in \( (0, 1) \) and \( \{ r_n \} \subset (0, \infty) \) is a real sequence satisfying the following conditions:

(i) \( \lim_{n \to \infty} \alpha_n = 0 \) and \( \sum_{n=0}^{\infty} \alpha_n = \infty \);

(ii) \( 0 < \lim \inf_{n \to \infty} \beta_n \leq \lim \sup_{n \to \infty} \beta_n < 1 \);

(iii) \( 0 < \lim \inf_{n \to \infty} \gamma_n \leq \lim \sup_{n \to \infty} \gamma_n < 1 \) and \( \lim_{n \to \infty} |\gamma_{n+1} - \gamma_n| = 0 \);

(iv) \( 0 < \lim \inf_{n \to \infty} r_n \) and \( \lim_{n \to \infty} |r_{n+1} - r_n| = 0 \).

Let \( \{ f_n \} \) be a sequence of \( \rho_n \)-contractive self-maps of \( C \) with

\[
0 \leq \rho_l = \lim \inf_{n \to \infty} \rho_n \leq \lim \sup_{n \to \infty} \rho_n = \rho_u < 1.
\]

Assume \( x_0 \in C \), \( 0 < \gamma < \frac{\alpha}{\rho_u} \) where \( \gamma \) is some constant, \( \{ f_n(x) \} \) is uniformly convergent for any \( x \in D \), where \( D \) is any bounded subset of \( C \) and \( \{ \lambda_n \}_{n=1}^{\infty} \) is a sequence of positive numbers in \([0, b]\) for some \( b \in (0, 1) \). If one define \( f(x) := \lim_{n \to \infty} f_n(x) \) for all \( x \in C \), then the sequences \( \{ x_n \} \) and \( \{ u_n \} \) generated iteratively by

\[
\begin{aligned}
\phi(u_n, x) + \frac{1}{\lambda_n} (x - u_n, u_n - x_n) &\geq 0 \text{ for all } x \in C, \\
y_n &= (1 - \gamma_n)x_n + \gamma_n W_n u_n, \\
x_{n+1} &= \alpha_n \gamma f_n(y_n) + \beta_n x_n + (1 - \beta_n) I - \alpha_n A W_n y_n,
\end{aligned}
\]

converge strongly to \( x^* \in F \), where \( x^* = P_{\bigcap_{n=1}^{\infty} F(T_n) \cap EP(\phi)}(I - A + \gamma f)(x^*) \).

Proof. Let \( Q = P_F \). Then

\[
\| Q(I - A + \gamma f)(x) - Q(I - A + \gamma f)(y) \|
\leq \| (I - A + \gamma f)(x) - (I - A + \gamma f)(y) \|
\leq \| (I - A)(x) - (I - A)(y) \| + \gamma\| f(x) - f(y) \|
\leq (1 - \gamma)\| x - y \| + \gamma\| f(x) - f(y) \|
= (1 - (\gamma - \gamma\alpha))\| x - y \|
\]

for all \( x, y \in F \). Therefore, \( Q(I - A + \gamma f) \) is a contraction of \( F \) into itself. So, there exists a unique element \( x^* \in F \) such that \( x^* = Q(I - A + \gamma f)(x^*) = P_{\bigcap_{n=1}^{\infty} F(T_n) \cap EP(\phi)}(I - A + \gamma f)(x^*) \). Note that from the condition (i), we may assume, without loss of generality, \( \alpha_n \leq (1 - \beta_n)\| A \|^{-1} \). Since \( A \) is strongly positive bounded linear operator on \( H \), we have

\[
\| A \| = \sup\{\| \langle Ax, x \rangle \| : x \in H, \| x \| = 1 \}.
\]

125
Observe that
\[
\langle (1 - \beta_n)I - \alpha_n A, x \rangle = (1 - \beta_n) - \alpha_n \langle Ax, x \rangle \\
\geq 1 - \beta_n - \alpha_n \|A\| \geq 0,
\]
that is to say \((1 - \beta_n)I - \alpha_n A\) is positive. It follows that
\[
\| (1 - \beta_n)I - \alpha_n A \| \\
= \sup \{ \langle (1 - \beta_n)I - \alpha_n A, x \rangle : x \in H, \| x \| = 1 \} \\
= \sup \{ 1 - \beta_n - \alpha_n \langle Ax, x \rangle : x \in H, \| x \| = 1 \} \\
\leq 1 - \beta_n - \alpha_n \gamma.
\]
Let \(p \in F\). From the definition of \(T_r\), we know that \(u_n = T_r x_n\). It follows that
\[
\| u_n - p \| = \| T_r x_n - T_r p \| \leq \| x_n - p \|,
\]
and hence
\[
\| y_n - p \| = \| (1 - \gamma_n)(x_n - p) + \gamma_n (W_n u_n - p) \| \\
\leq (1 - \gamma_n) \| x_n - p \| + \gamma_n \| W_n u_n - p \| \\
\leq (1 - \gamma_n) \| x_n - p \| + \gamma_n \| u_n - p \| \\
\leq (1 - \gamma_n) \| x_n - p \| + \gamma_n \| x_n - p \| = \| x_n - p \|.
\]
First, we claim that \(\{x_n\}\) and \(\{y_n\}\) are bounded. Indeed, from (4), (3) and (6), we obtain
\[
\| x_{n+1} - p \| \\
= \| \alpha_n (\gamma f_n(y_n) - Ap) + \beta_n (x_n - p) + ((1 - \beta_n)I - \alpha_n A)(W_n y_n - p) \| \\
\leq (1 - \beta_n - \alpha_n \gamma) \| y_n - p \| + \beta_n \| x_n - p \| + \alpha_n \| \gamma f_n(y_n) - Ap \| \\
\leq (1 - \alpha_n \gamma) \| x_n - p \| + \alpha_n \gamma \| f_n(y_n) - f_n(p) \| + \alpha_n \| \gamma f_n(p) - Ap \| \\
\leq (1 - \alpha_n (\gamma - \rho_n \gamma)) \| x_n - p \| + \alpha_n \| \gamma f_n(p) - Ap \|.
\]
By induction, \(\| x_n - p \| \leq \max \{ \| x_0 - p \|, \frac{1}{1 - \rho_n \gamma} \| \gamma f_n(p) - Ap \| \}, \quad n \geq 1\). From the uniform convergence of \(\{f_n\}\) on any bounded subset of \(C\), we conclude \(\{f_n(p)\}\) is bounded. Hence \(\{x_n\}\) is bounded, so are \(\{u_n\}\), \(\{y_n\}\), \(\{f_n(y_n)\}\), \(\{W_n u_n\}\) and \(\{W_n y_n\}\).
Define
\[
x_{n+1} = (1 - \beta_n)z_n + \beta_n x_n, \quad n \geq 0.
\]
Then
\[ z_{n+1} - z_n = \frac{x_{n+2} - \beta_{n+1} x_{n+1}}{1 - \beta_{n+1}} - \frac{x_{n+1} - \beta_n x_n}{1 - \beta_n} \]
\[ = \frac{\alpha_{n+1} \gamma f_n(y_{n+1}) + ((1 - \beta_{n+1}) - \alpha_{n+1} A) W_{n+1} y_{n+1}}{1 - \beta_{n+1}} \]
\[ - \frac{\alpha_n \gamma f_n(y_n) + ((1 - \beta_n) - \alpha_n A) W_n y_n}{1 - \beta_n} \]
\[ = \frac{\alpha_{n+1}}{1 - \beta_{n+1}} \gamma f_n(y_{n+1}) - \frac{\alpha_n}{1 - \beta_n} \gamma f_n(y_n) + W_{n+1} y_{n+1} + W_n y_n \]
\[ \leq (1 - \gamma_{n+1}) x_{n+1} + \gamma_{n+1} W_{n+1} u_{n+1} - (1 - \gamma_n) x_n - \gamma_n W_n u_n \]
\[ \leq (1 - \gamma_{n+1}) ||x_{n+1} - x_n|| + ||x_n|| + ||W_{n+1} u_{n+1} - W_n u_n|| \]
\[ + ||\gamma_{n+1} - \gamma_n|| ||W_n u_n|| \]
\[ \leq (1 - \gamma_{n+1}) ||x_{n+1} - x_n|| + ||x_n|| + ||W_{n+1} u_{n+1} - W_n u_n|| + ||W_{n+1} u_n - W_n u_n|| \]
\[ + ||\gamma_{n+1} - \gamma_n|| ||W_n u_n|| \]
\[ \leq (1 - \gamma_{n+1}) ||x_{n+1} - x_n|| + ||x_n|| \]
\[ + ||W_{n+1} u_{n+1} - W_n u_n|| + ||W_{n+1} u_n - W_n u_n|| \]
From (4), Since \( T_i \) and \( U_{n,i} \) are nonexpansive, we have for each \( n \geq 1 \)
\[ ||W_{n+1} u_n - W_n u_n|| = \frac{||\lambda_1 T_1 U_{n+1,2} u_n - \lambda_1 T_1 U_{n,2} u_n||}{\lambda_1} \]
\[ \leq \frac{||U_{n+1,2} u_n - U_{n,2} u_n||}{\lambda_1} \]
\[ = \frac{||\lambda_2 T_2 U_{n+1,3} u_n - \lambda_2 T_2 U_{n,3} u_n||}{\lambda_1} \]
\[ \leq \frac{||U_{n+1,3} u_n - U_{n,3} u_n||}{\lambda_1 \lambda_2 \ldots \lambda_n} \]
\[ \leq \ldots \]
\[ \leq \lambda_1 \lambda_2 \ldots \lambda_n ||W_{n+1,1} u_n - W_{n,1} u_n|| \]
\[ \leq M \prod_{i=1}^{n} \lambda_i, \]
and similarly
\[ ||W_{n+1} u_n - W_n u_n|| \leq \lambda_1 \lambda_2 \ldots \lambda_n ||W_{n+1,1} u_n - W_{n,1} u_n|| \leq M \prod_{i=1}^{n} \lambda_i, \]

127
for some constant $M \geq 0$. On the other hand, from $u_n = T_n x_n$ and $u_{n+1} = T_{n+1} x_{n+1},$

$$\phi(u_n, y) + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq 0, \quad \text{for all } y \in C, \quad (12)$$

and

$$\phi(u_{n+1}, y) + \frac{1}{r_{n+1}} \langle y - u_{n+1}, u_{n+1} - x_{n+1} \rangle \geq 0, \quad \text{for all } y \in C. \quad (13)$$

Putting $y = u_{n+1}$ in (12) and $y = u_n$ in (13), we obtain

$$\phi(u_n, u_{n+1}) + \frac{1}{r_n} \langle u_{n+1} - u_n, u_n - x_n \rangle \geq 0,$$

and

$$\phi(u_{n+1}, u_n) + \frac{1}{r_{n+1}} \langle u_n - u_{n+1}, u_{n+1} - x_{n+1} \rangle \geq 0.$$ So, from (A2)

$$\langle u_{n+1} - u_n, \frac{u_n - x_n}{r_n} - \frac{u_{n+1} - x_{n+1}}{r_{n+1}} \rangle \geq 0,$$

and hence

$$\langle u_{n+1} - u_n, u_n - u_{n+1} + u_{n+1} - x_n - \frac{r_n}{r_{n+1}} (u_{n+1} - x_{n+1}) \rangle \geq 0.$$ Without loss of generality, we may assume that there exists a real number $r$ such that $0 < r < r_n$ for all $n \geq 0.$ Therefore

$$\|u_{n+1} - u_n\|^2 \leq \langle u_{n+1} - u_n, x_{n+1} - x_n + (1 - \frac{r_n}{r_{n+1}}) (u_{n+1} - x_{n+1}) \rangle \leq \|u_{n+1} - u_n\| \{ \|x_{n+1} - x_n\| + |1 - \frac{r_n}{r_{n+1}}| \|u_{n+1} - x_{n+1}\| \}. \quad (14)$$

So

$$\|u_{n+1} - u_n\| \leq \|x_{n+1} - x_n\| + |1 - \frac{r_n}{r_{n+1}}| \|u_{n+1} - x_{n+1}\| \leq \|x_{n+1} - x_n\| + \frac{1}{r} |r_n - r_{n+1}| L,$$

where $L = \sup \{ \|u_n - x_n\| : n \geq 0 \}.$ Substituting (10) and (14) in (9), we have

$$\leq (1 - \gamma_{n+1}) \|x_{n+1} - x_n\| + |\gamma_{n+1} - \gamma_n| \|x_n\| + \gamma_{n+1} \|x_{n+1} - x_n\| + \frac{1}{r} |r_n - r_{n+1}| L + \gamma_{n+1} M \prod_{i=1}^{n} \lambda_i + |\gamma_{n+1} - \gamma_n| \|W_n u_n\| \leq (1 - \gamma_{n+1}) \|x_{n+1} - x_n\| + \gamma_{n+1} \|x_{n+1} - x_n\| + \frac{1}{r} |r_n - r_{n+1}| L + \gamma_{n+1} M \prod_{i=1}^{n} \lambda_i + |\gamma_{n+1} - \gamma_n| \|W_n u_n\|. \quad (15)$$

128
Combining (8), (11) and (15), we obtain

\[
\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\| \\
\leq \frac{\alpha_{n+1}}{1 - \beta_{n+1}} \left( \|\gamma f_n(y_{n+1})\| + \|AW_{n+1}y_{n+1}\| \right) \\
+ \frac{\alpha_n}{1 - \beta_n} \left( \|AW_ny_n\| + \|\gamma f_n(y_n)\| \right) \\
+ \|W_{n+1}y_{n+1} - W_ny_n\| + \|W_{n+1}y_n - W_ny_n\| \\
- \|x_{n+1} - x_n\| \\
\leq \frac{\alpha_{n+1}}{1 - \beta_{n+1}} \left( \|\gamma f_n(y_{n+1})\| + \|AW_{n+1}y_{n+1}\| \right) \\
+ \frac{\alpha_n}{1 - \beta_n} \left( \|AW_ny_n\| + \|\gamma f_n(y_n)\| \right) \\
+ \left[ \|x_{n+1} - x_n\| + |\gamma_{n+1} - \gamma_n| \|x_n\| \right] \\
+ \frac{1}{p} |r_n - r_{n+1}|L + M \prod_{i=1}^n \lambda_i + |\gamma_{n+1} - \gamma_n| \|W_nu_n\| + M \prod_{i=1}^n \lambda_i - \|x_{n+1} - x_n\| \\
\leq \frac{\alpha_{n+1}}{1 - \beta_{n+1}} \left( \|\gamma f_n(y_{n+1})\| + \|AW_{n+1}y_{n+1}\| \right) \\
+ \frac{\alpha_n}{1 - \beta_n} \left( \|AW_ny_n\| + \|\gamma f_n(y_n)\| \right) \\
+ |\gamma_{n+1} - \gamma_n| \|x_n\| + \frac{1}{p} |r_n - r_{n+1}|L \\
+ |\gamma_{n+1} - \gamma_n| \|W_nu_n\| + 2M \prod_{i=1}^n \lambda_i.
\]

Thus it follows from (16) and condition (i) – (iv) that (noting that 0 < \lambda_i \leq b < 1 for all i \geq 1)

\[
\limsup_{n \to \infty} (\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \leq 0.
\]

Hence by Lemma 6, we have \lim_{n \to \infty} \|z_n - x_n\| = 0. Consequently

\[
\lim_{n \to \infty} \|x_{n+1} - x_n\| = \lim_{n \to \infty} \left( 1 - \beta_n \right) \|z_n - x_n\| = 0.
\]

From (14) and \lim_{n \to \infty} |r_{n+1} - r_n| = 0, \lim_{n \to \infty} \|u_{n+1} - u_n\| = 0. From (5),

\[
\|x_n - W_ny_n\| \leq \|x_{n+1} - x_n\| + \alpha_n \|\gamma f_n(y_n) - AW_ny_n\| + \beta_n \|x_n - W_ny_n\|.
\]
That is \( \|x_n - W_n y_n\| \leq \alpha_n \|x_{n+1} - x_n\| + \frac{\alpha_n}{1 - \beta_n} \|\gamma f_n(y_n) - AW_n y_n\| \). It follows that
\[
\lim_{n \to \infty} \|x_n - W_n y_n\| = 0. \tag{17}
\]

For \( p \in F \), since \( T_r \) is firmly nonexpansive, we have
\[
\|u_n - p\|^2 = \|T_r x_n - T_r p\|^2 \leq \langle T_r x_n - T_r p, x_n - p \rangle
\]
\[
= \langle u_n - p, x_n - p \rangle = \frac{1}{2} (\|u_n - p\|^2 + \|x_n - p\|^2 - \|x_n - u_n\|^2),
\]
and hence \( \|u_n - p\|^2 \leq \|x_n - p\|^2 - \|x_n - u_n\|^2 \). Therefore
\[
\|x_{n+1} - p\|^2 = \|x_n - p\|^2 - \|x_n - u_n\|^2 + \|x_n - p\|^2 + \|u_n - p\|^2
\]
\[
= \|\alpha_n \gamma f_n(y_n) + \beta_n x_n + ((1 - \beta_n)I - \alpha_n A)W_n y_n - p\|^2
\]
\[
= \|\gamma f_n(y_n) - AW_n y_n\|^2 + \|\beta_n(x_n - p)\|^2 + \alpha_n \|\gamma f_n(y_n)\|^2
\]
\[
= \alpha_n^2 \|\gamma f_n(y_n) - AW_n y_n\|^2 + \|\beta_n(x_n - p)\|^2 + \alpha_n \|\gamma f_n(y_n) - AW_n y_n\|^2
\]
\[
\leq (1 - \beta_n)\|y_n - p\|^2 + \beta_n\|x_n - p\|^2 + \alpha_n^2 \|\gamma f_n(y_n) - AW_n y_n\|^2
\]
\[
+2\alpha_n(1 - \beta_n)\|W_n y_n - p, \gamma f_n(y_n) - AW_n y_n\|
\]
\[
+2\alpha_n\beta_n\|x_n - p, \gamma f_n(y_n) - AW_n y_n\|
\]
Thus
\[ (1 - \beta_n)\gamma_n \|x_n - u_n\|^2 \leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + \alpha_n^2\|\gamma f_n(y_n) - AW_n y_n\|^2 \]
\[ + 2\alpha_n \|x_n - p\|\|\gamma f_n(y_n) - AW_n y_n\| \]
\[ = (\|x_n - p\| - \|x_{n+1} - p\|)(\|x_n - p\| + \|x_{n+1} - p\|) \]
\[ + \alpha_n^2\|\gamma f_n(y_n) - AW_n y_n\|^2 \]
\[ + 2\alpha_n \|x_n - p\|\|\gamma f_n(y_n) - AW_n y_n\|. \]
Since \( \liminf_{n \to \infty} (1 - \beta_n) > 0 \) and \( \liminf_{n \to \infty} \gamma_n > 0 \), it is easy to see that \( \liminf_{n \to \infty} (1 - \beta_n)\gamma_n > 0 \). So
\[ \lim_{n \to \infty} \|x_n - u_n\| = 0. \] (18)
Observe that
\[ \|y_n - u_n\| \leq \|y_n - x_n\| + \|x_n - u_n\| \]
\[ \leq \gamma_n\|W_n u_n - x_n\| + \|x_n - u_n\| \]
\[ \leq \gamma_n\|W_n u_n - W_n y_n + W_n y_n - x_n\| + \|x_n - u_n\| \]
\[ \leq \gamma_n[\|y_n - u_n\| + \|W_n y_n - x_n\|] + \|x_n - u_n\|, \]
and hence \( (1 - \gamma_n)\|y_n - u_n\| \leq \|W_n y_n - x_n\| + \|x_n - u_n\| \). So, from (17), (18) and \( \limsup_{n \to \infty} \gamma_n < 1 \),
\[ \lim_{n \to \infty} \|y_n - u_n\| = 0 \] (19)
and so \( \lim_{n \to \infty} \|x_n - y_n\| = 0 \). Since
\[ \|W_n u_n - u_n\| \leq \|W_n u_n - W_n y_n\| + \|W_n y_n - x_n\| + \|x_n - u_n\| \]
\[ \leq \|y_n - u_n\| + \|W_n y_n - x_n\| + \|x_n - u_n\|, \]
we also have \( \lim_{n \to \infty} \|W_n u_n - u_n\| = 0 \). On the other hand, observe that
\[ \|W u_n - u_n\| \leq \|W_n u_n - W u_n\| + \|W_n u_n - u_n\|. \] (20)
It follows from (20) and Remark 2, we obtain \( \lim_{n \to \infty} \|W u_n - u_n\| = 0 \).

Next, we claim that
\[ \limsup_{n \to \infty} (\gamma f(x^*) - Ax^*, x_n - x^*) \leq 0, \] (21)
where \( x^* = P_{F(W)} \cap EP(\phi)(I - A + \gamma f)x^* \). First, we can choose a subsequence \( \{u_{n_j}\} \) of \( \{u_n\} \) such that
\[ \lim_{j \to \infty} (\gamma f(x^*) - Ax^*, u_{n_j} - x^*) = \limsup_{n \to \infty} (\gamma f(x^*) - Ax^*, u_n - x^*). \]

Finally, we prove that 

\[ x \in \text{EP}(\phi) \quad \text{and so} \quad \phi(y,w) \leq 0 \]  

for all \( y \in C \).

For \( t \) with \( 0 < t \leq 1 \) and \( y \in C \), let \( y_t = ty + (1-t)w \). Since \( y \in C \) and \( w \in C \), we have \( y_t \in C \) and hence \( \phi(y_t,w) \leq 0 \). So, from (A1) and (A4),

\[
0 = \phi(y_t,y_t) \leq t \phi(y_t,y) + (1-t) \phi(y_t,w) \leq t \phi(y_t,y),
\]

and so \( \phi(y_t,y) \geq 0 \). From (A3), \( \phi(w,y) \geq 0 \) for all \( y \in C \), and hence \( w \in \text{EP}(\phi) \).

Next, we show \( w \in F(W) \). Assume \( w \notin F(W) \). Since \( u_n \to w \) and \( Ww \neq w \), from Lemma 9 we have

\[
\liminf_{j \to \infty} \| u_{n_j} - w \| < \liminf_{j \to \infty} \| u_{n_j} - Ww \| \leq \liminf_{j \to \infty} (\| u_{n_j} - Ww \| + \| Ww - Ww \|) \leq \liminf_{j \to \infty} \| u_{n_j} - w \|.
\]

This is a contradiction. So, \( w \in F(W) = \bigcap_{n=1}^{\infty} F(T_n) \). Therefore, \( w \in F \). Since \( x^* = P_{\bigcap_{n=1}^{\infty} F(T_n) \cap \text{EP}(\phi)} (I - A + \gamma f)x^* \), we obtain

\[
\limsup_{n \to \infty} \langle \gamma f(x^*) - Ax^*, x_n - x^* \rangle 
\]

\[
= \lim_{j \to \infty} \langle \gamma f(x^*) - Ax^*, x_{n_j} - x^* \rangle 
\]

\[
= \lim_{j \to \infty} \langle \gamma f(x^*) - Ax^*, u_{n_j} - x^* \rangle 
\]

\[
= \langle \gamma f(x^*) - Ax^*, w - x^* \rangle \leq 0.
\]

From (17),

\[
\limsup_{n \to \infty} \langle \gamma f(x^*) - Ax^*, W_n y_n - x^* \rangle = \limsup_{n \to \infty} \langle \gamma f(x^*) - Ax^*, x_n - x^* \rangle \leq 0. \quad (22)
\]

Finally, we prove that \( \{ x_n \} \) converges strongly to \( x^* = P_{\text{EP}(\phi)} (I - A + \gamma f)x^* \).
Indeed, from (3),

\[\begin{align*}
\|x_{n+1} - x^*\|^2 &= \|\alpha_n (\gamma f_n(y_n) - Ax^*) + \beta_n (x_n - x^*)
+ ((1 - \beta_n) I - \alpha_n A)(W_n y_n - x^*)\|^2 \\
&= \alpha_n^2 \|\gamma f_n(y_n) - Ax^*\|^2 + \beta_n (x_n - x^*)
+ ((1 - \beta_n) I - \alpha_n A)(W_n y_n - x^*)\|^2 \\
&+ 2\beta_n \langle x_n - x^*, \gamma f_n(y_n) - Ax^* \rangle \\
&+ 2\alpha_n \langle (1 - \beta_n)(I - \alpha_n A)(W_n y_n - x^*) + \gamma f_n(y_n) - Ax^* \rangle \\
&\leq ((1 - \beta_n - \alpha_n \bar{\gamma}) ||W_n y_n - x^*|| + \beta_n ||x_n - x^*|| \|^2 \\
&+ \alpha_n^2 ||\gamma f_n(y_n) - Ax^*||^2 + 2\beta_n \alpha_n \gamma (x_n - x^*, f_n(y_n) - f_n(x^*)) \\
&+ 2\beta_n \alpha_n \langle x_n - x^*, \gamma f_n(x^*) - Ax^* \rangle \\
&+ 2(1 - \beta_n) \gamma \alpha_n \langle W_n y_n - x^*, f_n(y_n) - f_n(x^*) \rangle \\
&+ 2(1 - \beta_n) \alpha_n \langle W_n y_n - x^*, \gamma f_n(x^*) - Ax^* \rangle \\
&- 2\alpha_n^2 \langle A(W_n y_n - x^*), \gamma f_n(y_n) - Ax^* \rangle, \\
\end{align*}\]

Which implies that

\[\begin{align*}
\|x_{n+1} - x^*\|^2 &\leq [(1 - \alpha_n \bar{\gamma})^2 + 2\rho_n \beta_n \alpha_n \gamma + 2\beta_n (1 - \beta_n) \alpha_n \gamma] \|x_n - x^*\|^2 \\
&+ 2\beta_n \alpha_n \langle x_n - x^*, \gamma f_n(x^*) - Ax^* \rangle \\
&+ \alpha_n^2 \|\gamma f_n(y_n) - Ax^*\|^2 \\
&+ 2(1 - \beta_n) \alpha_n \langle W_n y_n - x^*, \gamma f_n(x^*) - Ax^* \rangle \\
&- 2\alpha_n^2 \langle A(W_n y_n - x^*), \gamma f_n(y_n) - Ax^* \rangle \\
&\leq [1 - 2\alpha_n (\bar{\gamma} - \rho_n \gamma)] \|x_n - x^*\|^2 + \alpha_n^2 \|x_n - x^*\|^2 \\
&+ 2\beta_n \alpha_n \langle x_n - x^*, \gamma f_n(x^*) - Ax^* \rangle + \alpha_n^2 \|\gamma f_n(y_n) - Ax^*\|^2 \\
&+ 2(1 - \beta_n) \alpha_n \langle W_n y_n - x^*, \gamma f_n(x^*) - Ax^* \rangle \\
&+ 2\alpha_n^2 \langle \gamma f_n(y_n) - Ax^*\| \|A(W_n y_n - x^*)\| \\
&= [1 - 2\alpha_n (\bar{\gamma} - \rho_n \gamma)] \|x_n - x^*\|^2 + \alpha_n \{\alpha_n (\bar{\gamma}^2 \|x_n - x^*\|^2 \\
&+ \|\gamma f_n(y_n) - Ax^*\|^2 + 2\|\gamma f_n(y_n) - Ax^*\|| \|A(W_n y_n - x^*)\|) \\
&+ 2\beta_n \langle x_n - x^*, \gamma f_n(x^*) - Ax^* \rangle \\
&+ 2(1 - \beta_n) \langle W_n y_n - x^*, \gamma f_n(x^*) - Ax^* \rangle}. \\
\end{align*}\]

By Schwartz inequality,

\[\limsup_{n \to \infty} \langle x_n - x^*, \gamma f_n(x^*) - Ax^* \rangle \leq \lim_{n \to \infty} \gamma \|x_n - x^*\| \|f_n(x^*) - f(x^*)\| \\
+ \limsup_{n \to \infty} \langle x_n - x^*, \gamma f(x^*) - Ax^* \rangle.\]

From (21),

\[\limsup_{n \to \infty} \langle x_n - x^*, \gamma f_n(x^*) - Ax^* \rangle \leq 0.\]
Since \( \{x_n\}, \{f_n(y_n)\} \) and \( \{W_n y_n\} \) are bounded, we can take a constant \( M_1 \geq 0 \) such that 
\[
\bar{\gamma}^2 \|x_n - x^*\|^2 + \|f_n(y_n) - Ax^*\|^2 + 2\|\gamma f_n(y_n) - Ax^*\| \|A(W_n y_n - x^*)\| \leq M_1,
\]
for all \( n \geq 0 \). From (23),
\[
\|x_{n+1} - x^*\|^2 \leq [1 - 2\alpha_n(\bar{\gamma} - \rho_n\gamma)]\|x_n - x^*\|^2 + \alpha_n\xi_n,
\]
(25)
where \( \xi_n = 2\beta_n\|x_n - x^*\|\|f_n(x^*) - Ax^*\| + 2(1 - \beta_n)\|W_n y_n - x^*\|, \gamma f_n(x^*) - Ax^*\|^2 + \alpha_n M_1 \). By (i), (22) and (24), we get \( \lim \sup_{n \to \infty} \xi_n \leq 0 \). Now applying Lemma 7 to (25) concludes that \( x_n \to x^* \) as \( n \to \infty \). This completes the proof.

Taking \( f_n = f \) for all \( n \in \mathbb{N} \) where \( f \) is a contraction on \( C \) into itself in Theorem 12, we get

**Remark 3.** Theorem 12 is a generalization of [16, Theorem 2.11].

**Remark 4.** Let \( T_n x = x \) for all \( n \in \mathbb{N} \) and for all \( x \in C \) in (4). Then, \( W_n x = x \) for all \( x \in C \) in Theorem 12. Therefore, Theorem 12 is a generalization of [16, Corollary 2.12].

**Remark 5.** Let \( \phi(x,y) = 0 \) for all \( x,y \in C \) and \( r_n = 1 \) in Theorem 12, then Theorem 12 is a generalization of [16, Corollary 2.13].

**Remark 6.** Let \( A = I \) (identity map) with constant \( \bar{\gamma} = 1, \gamma = 1 \) and \( \eta_n = 1 - \alpha_n - \beta_n \) in Theorem 12, then Theorem 12 is a generalization of [7, Theorem 3.1].

### 4. Numerical Test

In this section, we give an example to illustrate the scheme (5) given in Theorem 12.

**Example 3.1** Let \( C = [-1, 1] \subset H = \mathbb{R} \) and define \( \phi(x,y) = -5x^2 + xy + 4y^2 \). It is easy to see verify that \( \phi \) satisfies the conditions \( (A_1) - (A_4) \). From Lemma 2.2, \( T_r \) is single-valued for all \( r > 0 \). Now, we deduce a formula for \( T_r(x) \). For any \( y \in [-1, 1] \) and \( r > 0 \), we have
\[
\phi(z,y) + \frac{1}{r}\langle y - z, z - x \rangle \geq 0 \iff 4ry^2 + ((r + 1)z - x)y + xz - (5r + 1)z^2 \geq 0.
\]

Set \( G(y) = 4ry^2 + ((r + 1)z - x)y + xz - (5r + 1)z^2 \). Then \( G(y) \) is a quadratic function of \( y \) with coefficients \( a = 4r, b = (r + 1)z - x \) and \( c = xz - (5r + 1)z^2 \). So its discriminate \( \Delta = b^2 - 4ac \) is
\[
\Delta = ((r + 1)z - x)^2 - 16r(xz - (5r + 1)z^2)
\]
\[
= (r + 1)^2z^2 - 2(r + 1)xz + x^2 - 16rxz + (80r^2 + 16r)z^2
\]
\[
= [(9r + 1)z - x]^2.
\]
Since $G(y) \geq 0$ for all $y \in C$, this is true if and only if $\Delta \leq 0$. That is, $[(9r+1)z-x]^2 \leq 0$. Therefore, $z = \frac{x}{9r+1}$, which yields $T_r(x) = \frac{x}{9r+1}$. So, from Lemma 3, we get $\text{EP}(\phi) = \{0\}$. Let $\alpha_n = \frac{1}{n}, \beta_n = \frac{n}{3n+1}, \lambda_n = \beta \in (0,1), \gamma_n = \frac{1}{2}, r_n = 1, T_n = I$, for all $n \in \mathbb{N}$, $Ax = x$ with coefficient $\gamma = 1$, $f_n(x) = \frac{n}{3n+1}x$ and $\gamma = \frac{1}{2}$. Hence, $F = \bigcap_{n=1}^{\infty} F(T_n) \cap \text{EP}(\phi) = \{0\}$. Also, $W_n = I$. Indeed, from (4), we have

$$W_1 = U_{1,1} = \lambda_1 T_1 U_{1,2} + (1-\lambda_1)I = \lambda_1 T_1 + (1-\lambda_1)I,$$

$$W_2 = U_{2,1} = \lambda_1 T_1 U_{2,2} + (1-\lambda_1)I = \lambda_1 T_1(\lambda_2 T_2 U_{2,3} + (1-\lambda_2)I)$$

$$= \lambda_1 \lambda_2 T_1 T_2 + \lambda_1 (1-\lambda_2)T_1 + (1-\lambda_1)I,$$

$$W_3 = U_{3,1} = \lambda_1 T_1 U_{3,2} + (1-\lambda_1)I = \lambda_1 T_1(\lambda_2 T_2 U_{3,3} + (1-\lambda_2)I)$$

$$+ (1-\lambda_1)I$$

$$= \lambda_1 \lambda_2 T_1 T_2 U_{3,3} + \lambda_1 (1-\lambda_2)T_1 + (1-\lambda_1)I$$

$$= \lambda_1 \lambda_2 T_1 T_2(\lambda_3 T_3 U_{3,4} + (1-\lambda_3)I) + \lambda_1 (1-\lambda_2)T_1$$

$$+ (1-\lambda_1)I$$

$$= \lambda_1 \lambda_2 \lambda_3 T_1 T_2 T_3 + \lambda_1 \lambda_2 (1-\lambda_3) T_1 T_2 + \lambda_1 (1-\lambda_2)T_1$$

$$+ (1-\lambda_1)I.$$

By computing in this way by (4), we obtain

$$W_n = U_{n,1} = \lambda_1 \lambda_2 \ldots \lambda_n T_1 T_2 \ldots T_n$$

$$+ \lambda_1 \lambda_2 \ldots \lambda_{n-1} (1-\lambda_n) T_1 T_2 \ldots T_{n-1}$$

$$+ \lambda_1 \lambda_2 \ldots \lambda_{n-2} (1-\lambda_n) T_1 T_2 \ldots T_{n-2}$$

$$+ \lambda_1 (1-\lambda_2) T_1 + (1-\lambda_1)I.$$

Since $T_n = I, \lambda_n = \beta$ for all $n \in \mathbb{N}$, we get

$$W_n = (\beta^n + \beta^{n-1} (1-\beta) + \ldots + \beta (1-\beta) + (1-\beta)) I = I.$$

Then, from Lemma 7, the sequences $\{x_n\}$ and $\{u_n\}$, generated iteratively by

$$\begin{aligned}
&\begin{cases}
    u_n = T_n x_n = \frac{1}{10} x_n, \\
    y_n = \frac{1}{2} x_n + \frac{1}{2} W_n u_n = \frac{11}{20} x_n,
\end{cases} \\
&x_{n+1} = \frac{84n^2 - 32n^2 - 22}{40n(3n+1)} x_n,
\end{aligned}$$

(26)

converges strongly to $0 \in F$, where $0 = P_F(\frac{1}{6}I)(0)$.

References


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