Generic lightlike submanifolds of an indefinite Kaehler manifold with an $(\ell, m)$-type connection

Dae Ho Jin and Chul Woo Lee

**Abstract.** We study generic lightlike submanifolds $M$ of an indefinite Kaehler manifold $M$ with an $(\ell, m)$-type connection subject to the condition that the characteristic vector field $\zeta$ of $M$ belongs to our screen distribution $S(TM)$ of $M$. We provide several new results on such a generic lightlike submanifold. Also, we investigate generic lightlike submanifolds of an indefinite complex space form $\tilde{M}(c)$ with a semi-symmetric metric connection subject such that $\zeta$ belongs to $S(TM)$.

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**Key words:** generic lightlike submanifold; semi-symmetric metric connection; indefinite Kaehler manifold; indefinite complex space form.

1 Introduction

This author introduced a non-symmetric and non-metric connection on semi-Riemannian manifolds $(M, \tilde{g})$ in paper [5] as follows:

A linear connection $\nabla$ on $(M, \tilde{g})$ is called an $(\ell, m)$-type connection if this connection $\nabla$ and its torsion tensor $\tilde{T}$ satisfy

\begin{align}
(\nabla_X \tilde{g})(Y, Z) &= -\ell\{\theta(Y)\tilde{g}(X, Z) + \theta(Z)\tilde{g}(X, Y)\} \\
&\quad - m\{\theta(Y)\tilde{g}(JX, Z) + \theta(Z)\tilde{g}(JX, Y)\}, \\
\tilde{T}(X, Y) &= \ell\{\theta(Y)X - \theta(X)Y\} + m\{\theta(Y)JX - \theta(X)JY\},
\end{align}

where $\ell$ and $m$ are smooth functions, $J$ is a tensor field of type $(1, 1)$ and $\theta$ is a 1-form associated with a smooth unit spacelike vector field $\zeta$, which is called the characteristic vector field, by $\theta(X) = \tilde{g}(X, \zeta)$. We set $(\ell, m) \neq (0, 0)$ and we denote by $\tilde{X}$, $\tilde{Y}$ and $\tilde{Z}$ the smooth vector fields on $\tilde{M}$.

**Remark 1.1.** Denote by $\tilde{\nabla}$ the Levi-Civita connection of a semi-Riemannian manifold $(M, \tilde{g})$ with respect to $\tilde{g}$. Then we see that a linear connection $\nabla$ on $M$ is an $(\ell, m)$-type connection if and only if $\nabla$ satisfies

\begin{equation}
\tilde{\nabla}_X \tilde{Y} = \tilde{\nabla}_X \tilde{Y} + \theta(\tilde{Y})\{\ell\tilde{X} + mJ\tilde{X}\}.
\end{equation}

A lightlike submanifold $M$ of an indefinite Kaehler manifold $(\tilde{M}, \tilde{g}, J)$, with an indefinite almost complex structure $J$, is called generic if there exists a screen distribution $S(TM)$, which is a complementary non-degenerate distribution of $Rad(TM) = TM \cap TM^\perp$ in $TM$, such that
\begin{equation}
J(S(TM)^\perp) \subset S(TM),
\end{equation}
where $S(TM)^\perp$ is the orthogonal complement of $S(TM)$ in the tangent bundle $T\tilde{M}$ of $\tilde{M}$ such that $TM = S(TM) \oplus_{\text{orth}} S(TM)^\perp$. The notion of generic lightlike submanifolds of indefinite almost complex manifolds or indefinite almost contact manifolds was introduced by Jin-Lee [6] and later, studied by several authors [2, 3, 4, 7].

The objective of study in this paper is generic lightlike submanifolds of an indefinite almost complex manifold $(\tilde{M}, g, J)$ with an $(\ell, m)$-type connection subject to the conditions that (1) the tensor field $J$, defined by (1.1) and (1.2), is identical with the indefinite almost complex structure tensor $J$ of $\tilde{M}$, and (2) the characteristic vector field $\zeta$ of $M$ belongs to $S(TM)$.

2 $(\ell, m)$-type connections

Let $\tilde{M} = (\tilde{M}, \tilde{g}, J)$ be an indefinite Kaehler manifold, where $\tilde{g}$ is a semi-Riemannian metric and $J$ is an indefinite almost complex structure;
\begin{equation}
J^2 \tilde{X} = -\tilde{X}, \quad \tilde{g}(J\tilde{X}, J\tilde{Y}) = \tilde{g}(\tilde{X}, \tilde{Y}), \quad (\tilde{\nabla}_X J)\tilde{Y} = 0.
\end{equation}
Replacing the Levi-Civita connection $\tilde{\nabla}$ by the $(\ell, m)$-type connection $\tilde{\nabla}$, the third equation of three equations in (2.1) is reduced to
\begin{equation}
(\tilde{\nabla}_X J)\tilde{Y} = \ell\{\theta(J\tilde{Y})\tilde{X} - \theta(\tilde{Y})J\tilde{X}\} + m\{\theta(\tilde{Y})\tilde{X} + \theta(J\tilde{Y})J\tilde{X}\}.
\end{equation}

Let $(M, g)$ be an $m$-dimensional lightlike submanifold of an indefinite Kaehler manifold $(\tilde{M}, \tilde{g}, J)$ of dimension $(m + n)$. Then the radical distribution $Rad(TM) = TM \cap TM^\perp$ is a subbundle of the tangent bundle $TM$ and the normal bundle $TM^\perp$, of rank $r$ ($1 \leq r \leq \min\{m, n\}$). In general, due to [1], we can take two complementary non-degenerate distributions $S(TM)$ and $S(TM^\perp)$ of $Rad(TM)$ in $TM$ and $TM^\perp$, respectively, which are called the screen distribution and the co-screen distribution of $M$, such that
\[ TM = Rad(TM) \oplus_{\text{orth}} S(TM), \quad TM^\perp = Rad(TM) \oplus_{\text{orth}} S(TM^\perp), \]
where $\oplus_{\text{orth}}$ denotes the orthogonal direct sum. Denote by $F(M)$ the algebra of smooth functions on $M$ and by $\Gamma(E)$ the $F(M)$ module of smooth sections of a vector bundle $E$ over $M$. Also denote by (2.1), the $i$-th equation of (2.1). We use the same notations for any others. Let $X, Y, Z$ and $W$ be the vector fields on $M$, unless otherwise specified. We use the following range of indices:
\[ i, j, k, \ldots \in \{1, \ldots, r\}, \quad a, b, c, \ldots \in \{r + 1, \ldots, n\}. \]

Let $tr(TM)$ and $ltr(TM)$ be complementary vector bundles to $TM$ in $TM|_{\mathcal{U}}$ and $TM^\perp$ in $S(TM)^\perp$, respectively, and let $\{N_1, \ldots, N_r\}$ be a null basis of $ltr(TM)|_{\mathcal{U}}$, where $\mathcal{U}$ is a coordinate neighborhood of $M$, such that
\[ \tilde{g}(N_i, \zeta_j) = \delta_{ij}, \quad \tilde{g}(N_i, N_j) = 0, \]
where \(\{\xi_1, \cdots, \xi_r\}\) is a null basis of \(\text{Rad}(TM)_{|\M} \). Then we have

\[
TM = TM \oplus \text{tr}(TM) = \{\text{Rad}(TM) \oplus \text{tr}(TM)\} \oplus_{\text{orth}} S(TM)
\]

\[
= \{\text{Rad}(TM) \oplus \text{ltr}(TM)\} \oplus_{\text{orth}} S(TM) \oplus_{\text{orth}} S(TM^\perp).
\]

A lightlike submanifold \(M = (M, g, S(TM), S(TM^\perp))\) of \(\M\) is called an \(r\)-lightlike submanifold [1] if \(1 \leq r < \min\{m, n\}\). For an \(r\)-lightlike \(M\), we see that \(S(TM) \neq \{0\}\) and \(S(TM^\perp) \neq \{0\}\). In the sequel, by saying that \(M\) is a lightlike submanifold we shall mean that it is an \(r\)-lightlike submanifold, with following local quasi-orthonormal field of frames of \(M\):

\[
\{\xi_1, \cdots, \xi_r, N_1, \cdots, N_r, F_{r+1}, \cdots, F_m, E_{r+1}, \cdots, E_n\},
\]

where \(\{F_{r+1}, \cdots, F_m\}\) and \(\{E_{r+1}, \cdots, E_n\}\) are orthonormal bases of \(S(TM)\) and \(S(TM^\perp)\), respectively. Denote \(\epsilon_a = \bar{g}(E_a, E_b)\). Then \(\epsilon_{a\beta b} = \bar{g}(E_a, E_b)\).

In this paper, we consider generic lightlike submanifolds \(M\) of an indefinite Kaehler manifold \(\M\) equipped with an \((\ell, m)\)-type connection and a screen distribution \(S(TM)\) which contains the characteristic vector field \(\zeta\). Let \(P\) be the projection morphism of \(TM\) on \(S(TM)\). Then the local Gauss and Weingarten formulae of \(M\) and \(S(TM)\) are given respectively by

\[
\nabla_X Y = \nabla_X Y + \sum_{i=1}^{r} h^i_\ell(X, Y)N_i + \sum_{a=r+1}^{n} h^a_\ell(X, Y)E_a,
\]

\[
\nabla_X N_i = -A_{N_i} X + \sum_{j=1}^{r} \tau_{ij}(X)N_j + \sum_{a=r+1}^{n} \rho_{ia}(X)E_a,
\]

\[
\nabla_X E_a = -A_{E_a} X + \sum_{i=1}^{r} \lambda_{ai}(X)N_i + \sum_{b=r+1}^{n} \mu_{ab}(X)E_b;
\]

\[
\nabla_X PY = \nabla_X PY + \sum_{i=1}^{r} h^i_\ell(X, PY)\xi_i,
\]

\[
\nabla_X \xi_i = -A^*_{\xi_i} X - \sum_{j=1}^{r} \tau_{ji}(X)\xi_j,
\]

where \(\nabla\) and \(\nabla^*\) are induced linear connections on \(M\) and \(S(TM)\), respectively, \(h^i_\ell\) and \(h^a_\ell\) are called the \textit{local second fundamental forms on} \(M\), \(h^*\) are called the \textit{local second fundamental forms on} \(S(TM)\). \(A_{N_i}, A_{E_a}\) and \(A^*_{\xi_i}\) are called the \textit{shape operators}, and \(\tau_{ij}, \rho_{ia}, \lambda_{ai}\) and \(\mu_{ab}\) are 1-forms on \(M\).

For a generic lightlike submanifold \(M\), from (1.4), we show that the distributions \(J(\text{Rad}(TM)), J(\text{ltr}(TM))\) and \(J(S(TM^\perp))\) are subbundles of \(S(TM)\). Thus there exist non-degenerate almost complex distributions \(H_o\) and \(H\) with respect to \(J\), i.e., \(J(H_o) = H_o\) and \(J(H) = H\), such that

\[
S(TM) = \{J(\text{Rad}(TM)) \oplus J(\text{ltr}(TM))\} \oplus_{\text{orth}} J(S(TM^\perp)) \oplus_{\text{orth}} H_o,
\]

\[
H = \text{Rad}(TM) \oplus_{\text{orth}} J(\text{Rad}(TM)) \oplus_{\text{orth}} H_o.
\]

In this case, the tangent bundle \(TM\) of \(M\) is decomposed as follow:

\[
TM = H \oplus J(\text{ltr}(TM)) \oplus_{\text{orth}} J(S(TM^\perp)).
\]
Consider \( r \)-th local null vector fields \( U_i \) and \( V_i \), \((n - r)\)-th local non-null unit vector fields \( W_a \), and their 1-forms \( u_i \), \( v_i \) and \( w_a \) defined by

\[
\begin{align*}
U_i &= -JN_i, & V_i &= -J\xi_i, & W_a &= -J\epsilon_a, \\
(2.9) & & \quad & w_a(X) &= \epsilon_ag(X, W_a).
\end{align*}
\]

Denote by \( S \) the projection morphism of \( TM \) on \( H \) and by \( F \) the tensor field of type (1, 1) globally defined on \( M \) by \( F = J \circ S \). Then \( JX \) is expressed as

\[
(2.11) \quad JX = FX + \sum_{i=1}^r u_i(X)N_i + \sum_{a=r+1}^n w_a(X)E_a.
\]

Applying \( J \) to (2.11) and using (2.1), (2.9) and (2.11), we have

\[
(2.12) \quad F^2X = -X + \sum_{i=1}^r u_i(X)U_i + \sum_{a=r+1}^n w_a(X)W_a.
\]

By using (2.1)\( _2 \) and (2.11), we obtain

\[
(2.13) \quad g(FX, FY) = g(X, Y) - \sum_{i=1}^r \{u_i(X)v_i(Y) + u_i(Y)v_i(X)\} - \sum_{a=r+1}^n \epsilon_aw_a(X)w_a(Y).
\]

We say that \( F \) is the structure tensor field of \( M \).

Using (1.1), (1.2, (2.3)) and (2.11), we see that

\[
(2.14) \quad (\nabla_X g)(Y, Z) = \sum_{i=1}^r \{h^i(X, Y)\eta_i(Z) + h^i(X, Z)\eta_i(Y)\} - \ell(\theta(Y)g(X,Z) + \theta(Z)g(X,Y)) \\
- m(\theta(Y)\tilde{g}(JX, Z) + \theta(Z)\tilde{g}(JX, Y)),
\]

\[
(2.15) \quad T(X, Y) = \ell(\theta(Y)X - \theta(Y)Y) + m(\theta(Y)FX - \theta(Y)FY),
\]

\[
(2.16) \quad h^i(X, Y) - h^i(Y, X) = m(\theta(Y)u_i(X) - \theta(X)u_i(Y)),
\]

\[
(2.17) \quad h^a(X, Y) - h^a(Y, X) = m(\theta(Y)w_a(X) - \theta(X)w_a(Y)),
\]

where \( \eta_i \)'s are 1-forms such that \( \eta_i(X) = \tilde{g}(X, N_i) \).

From the facts that \( h^i(X, Y) = \tilde{g}(\nabla_X Y, \xi_i) \) and \( \epsilon_a h^a(X, Y) = \tilde{g}(\nabla_X Y, E_a) \), we know that \( h^i \) and \( h^a \) are independent of the choice of \( S(TM) \). The above local second fundamental forms are related to their shape operators by

\[
(2.18) \quad h^i(X, Y) = g(A^i_{\xi_i} X, Y) - \sum_{k=1}^r h^i_k(X, \xi_i)\eta_k(Y) + m\theta(Y)u_i(X),
\]

\[
(2.19) \quad \epsilon_a h^a(X, Y) = g(A_{\xi_a} X, Y) - \sum_{k=1}^r \lambda_{ak}(X)\eta_k(Y) + \epsilon_am\theta(Y)w_a(X),
\]

\[
(2.20) \quad h^i(X, PY) = g(A_{\xi_i} X, PY) + \theta(PY)(\ell\eta_i(X) + mv_i(X)).
\]
Applying $\nabla_X$ to $g(E_a, E_b) = e\delta_{ab}$, $g(\xi_i, \xi_j) = 0$, $g(\xi_i, E_a) = 0$, $g(N_i, N_j) = 0$ and $g(N_i, E_a) = 0$ by turns, we obtain $e_a\mu_{ab} + e_a\mu_{ba} = 0$ and

\begin{equation}
(2.21) \quad h^a_i(X, \xi_j) + h^a_j(X, \xi_i) = 0, \quad h^a_i(X, \xi_j) = -e_a\lambda_{ai}(X), \\
\eta(X, N_i) + \eta(X, N_j) = 0, \quad \hat{g}(A_{e_a} X, N_i) = e_a\rho_{ia}(X).
\end{equation}

Furthermore, using (2.21) we see that

\begin{equation}
(2.22) \quad h^a_i(X, \xi_j) = 0, \quad h^a_j(X, \xi_i) = 0, \quad A^i_\xi \xi_i = 0.
\end{equation}

Applying $\nabla_X$ to (2.9), (2.3) and (2.11) by turns and using (2.2), (2.3)−(2.7), (2.10)−(2.12) and (2.9)−(2.11), we get

\begin{equation}
(2.23) \quad h^a_i(X, U_j) = u_j(A_{\xi_i}, X) + m\theta(U_i)u_j(X) \\
= h^a_i(X, V_j) + m\theta(U_i)u_j(X) \\
- \theta(V_j)\{\epsilon\eta(X) + mv_j(X)\},
\end{equation}

\begin{equation}
(2.24) \quad h^a_i(X, U_i) = w_a(A_{\xi_i}, X) + m\theta(U_i)w_a(X) \\
= e_a\epsilon h^a_i(X, W_a) + m\theta(U_i)w_a(X) \\
- e_a\theta(W_a)\{\epsilon\eta(X) + mv_i(X)\},
\end{equation}

\begin{equation}
(2.25) \quad h^a_i(X, V_i) = w_a(A_{\xi_i}, X) + m\theta(V_i)w_a(X) \\
= e_a\epsilon h^a_i(X, W_a) + m\theta(W_a)w_a(X) \\
- e_a\theta(W_a)\{\epsilon\eta(X) + mv_i(X)\},
\end{equation}

\begin{equation}
(2.26) \quad h^a_i(X, V_j) = h^a_i(X, V_j) + m\theta(V_j)u_j(X) - \theta(V_j)u_i(X),
\end{equation}

\begin{equation}
(2.27) \quad e_b h^a_i(X, W_b) - m\theta(W_a)w_b(X) \\
= e_a h^a_i(X, W_b) - m\theta(W_b)w_a(X),
\end{equation}

\begin{equation}
(2.28) \quad \nabla_X U_i = F(A_{\xi_i}, X) + \sum_{j=1}^{r} \tau_j(X)U_j + \sum_{a=r+1}^{n} \rho_a(X)W_a \\
+ \theta(U_i)\{\ell X + mFX\},
\end{equation}

\begin{equation}
(2.29) \quad \nabla_X V_i = F(A^i_\xi, X) - \sum_{j=1}^{r} \tau_j(X)V_j + \sum_{j=1}^{r} h^a_j(X, \xi_i)U_j \\
- \sum_{a=r+1}^{n} e_a\lambda_{ai}(X)W_a + \theta(V_i)\{\ell X + mFX\},
\end{equation}

\begin{equation}
(2.30) \quad \nabla_X W_a = F(A_{e_a} X) + \sum_{i=1}^{r} \lambda_{ai}(X)U_i + \sum_{b=r+1}^{n} \mu_{ab}(X)W_b, \\
+ \theta(W_a)\{\ell X + mFX\},
\end{equation}

\begin{equation}
(2.31) \quad (\nabla_X F)Y = \sum_{i=1}^{r} u_i(Y)A_{\xi_i} X + \sum_{a=r+1}^{n} w_a(Y)A_{e_a} X \\
- \sum_{i=1}^{r} h^a_i(X, Y)U_i - \sum_{a=r+1}^{n} h^a_i(X, Y)W_a \\
+ \ell\{\theta(FY)X - \theta(Y)FX\} + m\{\theta(Y)X + \theta(FY)FX\}. 
\end{equation}
Definition 2.1. We say that a lightlike submanifold $M$ of $\bar{M}$ is called
(1) irrotational [9] if $\nabla_X \xi_i \in \Gamma(TM)$ for all $i \in \{1, \cdots, r\}$,
(2) solenoidal [8] if $A_{E_a}$ and $A_{N_a}$ are $S(TM)$-valued,
(3) statical [8] if $M$ is both irrotational and solenoidal.

Remark 2.2. From (2.3) and (2.21), the item (1) is equivalent to
\begin{equation}
\sum_{k=1}^{r} h_{\ell}^a(Y, \xi_k) U_k + \sum_{a=r+1}^{n} h_{\sigma}^a(X, \xi_k) W_a = 0.
\end{equation}

By using (2.21), the item (2) is equivalent to
\begin{equation}
\sum_{k=1}^{r} h_{\ell}^a(Y, \xi_k) U_k + \sum_{a=r+1}^{n} h_{\sigma}^a(X, \xi_k) W_a = 0.
\end{equation}

3 Some results

Theorem 3.1. There exist no generic lightlike submanifold $M$ of an indefinite Kaehler
manifold $\bar{M}$ with an $(\ell, m)$-type connection such that $\xi$ belong to $S(TM)$ and $F$ is par-
allel with respect to the connection $\nabla$ on $\bar{M}$.

Proof. Taking the scalar product with $N_i$ to (2.31) with $\nabla_X F = 0$, we get
\begin{equation}
\sum_{k=1}^{r} u_k(Y) h_{\ell}(A_{N_k}X) + \sum_{a=r+1}^{n} w_a(Y) h_{\sigma}(A_{E_a}X)
+ (\ell h_{\ell}(X) + mv_i(X)) \theta(Y) - (\ell v_i(X) - m \eta_i(X)) \theta(Y) = 0.
\end{equation}

Replacing $Y$ by $\xi_j$ to (3.1) and using the fact that $F \xi_j = -V_j$, we have
\begin{equation}
\{\ell h_{\ell}(X) + mv_i(X)\} \theta(V_j) = 0.
\end{equation}

Taking $X = \xi_j$ and $X = V_j$ to this equation by turns, we obtain
\begin{equation}
\ell \theta(V_i) = 0, \quad m \theta(V_i) = 0 \quad \forall i.
\end{equation}

Replacing $Y$ by $\xi_j$ to (2.31) with $\nabla_X F = 0$ and using (3.2), we obtain
\begin{equation}
\sum_{k=1}^{r} h_{\ell}(X, \xi_j) U_k + \sum_{a=r+1}^{n} h_{\sigma}(X, \xi_j) W_a = 0.
\end{equation}

Taking the scalar product with $V_i$ and $W_b$ to this by turns, we have
\begin{equation}
\ell h_{\ell}(X, \xi_j) = 0, \quad m h_{\sigma}(X, \xi_j) = 0.
\end{equation}
Taking $Y = U_j$ to (3.1) and using the fact that $FU_j = 0$, we obtain

\begin{equation}
\eta_i(A_{X_j} X) = \{ \ell v_i(X) - m \eta_i(X) \} \theta(U_j).
\end{equation}

Taking $i = j$ to (3.4) and using (2.21), we obtain

\begin{equation}
\ell \theta(U_i) = 0,
\end{equation}

\begin{equation}
m \theta(U_i) = 0,
\end{equation}

\begin{equation}
\forall i.
\end{equation}

From (3.4) and (3.5), we see that

\begin{equation}
\eta_i(A_{X_j} X) = 0.
\end{equation}

Taking $Y = W_a$ to (2.31) with $\nabla_X F = 0$ and using (3.5), we obtain

\begin{equation}
A_{\ell} X = \sum_{i=1}^{r} h^i_{\ell} (X, W_a) U_i + \sum_{b=r+1}^{n} h^b_{\ell} (X, W_a) W_b
\end{equation}

\begin{equation}
+ \theta(W_a) \{ \ell F X + m X \}.
\end{equation}

Taking the scalar product with $U_i$ to (3.7) and using (2.19), we have

\begin{equation}
h^i_{\ell} (X, U_i) = -\epsilon_a \theta(W_a) \{ \ell \eta_i(X) + m v_i(X) \}.
\end{equation}

Taking $X = \xi_1$ to this, we have $h^i_{\ell} (\xi_1, U_i) = 0$. Also, taking $X = U_i$ to (3.3), we have $h^i_{\ell} (U_i, \xi_1) = 0$. Taking $X = U_i$ and $Y = \xi_1$ to (2.17), we have $h^i_{\ell} (U_i, \xi_1) = h^i_{\ell} (\xi_1, U_i)$. Therefore, we get $\ell \theta(W_a) = 0$. Taking the scalar product with $N_i$ to (3.7) and using $\ell \theta(W_a) = 0$, we obtain

\begin{equation}
\eta_i(A_{\ell} X) = -\epsilon_a m \theta(W_a) v_i(X).
\end{equation}

Replacing $X$ by $\xi_1$ to (3.1) and using (3.6) and (3.8), we have

\begin{equation}
\ell \theta(F Y) + m \theta(Y) = 0.
\end{equation}

Taking $Y = W_a$ to this, we have $m \theta(W_a) = 0$. Thus, from (3.8), we get

\begin{equation}
\eta_i(A_{\ell} X) = 0.
\end{equation}

Using (3.6), (3.9) and (3.10), the equation (3.1) is reduced to

\begin{equation}
m \theta(F Y) - \ell \theta(Y) = 0.
\end{equation}

As $(\ell, m) \neq (0, 0)$, from (3.9) and the last equation, we see that $\theta(X) = 0$, i.e., $g(\xi, X) = 0$ for all $X \in \Gamma(TM)$. As $\xi$ belongs to $S(TM)$, we see that $\xi = 0$. Hence $\theta = 0$. It is a contradiction to $\theta \neq 0$.

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**Theorem 3.2.** Let $M$ be a generic lightlike submanifold of an indefinite Kaehler manifold $M$ with an $(\ell, m)$-type connection such that $\zeta$ belongs to $S(TM)$. If $U_i$s are parallel with respect to the induced connection $\nabla$ of $M$, then $\tau_{ij} = 0$ for all $i$ and $j$, and $M$ is solenoidal.
Proof. Assume that $U_i$s are parallel with respect to the connection $\nabla$. Taking the scalar product with $U_j$ to (2.28) with $\nabla_X U_i = 0$, we have

$$\eta_j(A_{\eta_i} X) = \theta(U_i)\{\ell v_j(X) - m\eta_j(X)\}.$$  

Taking $i = j$ to this equation and using (2.21), we obtain

$$\theta(U_j)\{\ell v_j(X) - m\eta_j(X)\} = 0.$$  

Taking $X = V_i$ and $X = \xi_i$ to this equation, we have

$$\eta_j(A_{\eta_i} X) = 0.$$  

Taking the scalar product with $V_j$, $W_a$ and $N_j$ to (2.28) by turns, we have

$$\bar{\tau}_{ij} = 0, \quad \rho_{ia}(X) = \eta_i(A_{\eta_a} X) = 0, \quad h_i^*(X, U_j) = 0.$$  

From (3.11) and (3.12), we see that $\bar{\tau}_{ij} = 0$ and $M$ is solenoidal. \qed

**Theorem 3.3.** Let $M$ be a generic lightlike submanifold of an indefinite Kaehler manifold $\mathcal{M}$ with an $(\ell, m)$-type connection such that $\zeta$ belongs to $S(TM)$. If $V_i$s are parallel with respect to $\nabla$, then $M$ is irrotational.

**Proof.** Assume that $V_i$s are parallel with respect to $\nabla$. Taking the scalar product with $N_j$ to (2.29) with $\nabla_X V_i = 0$ and using (2.18), we have

$$h_i^*(X, U_j) = m\theta(U_j)u_i(X) - \theta(V_i)\{\ell v_j(X) + mv_j(X)\}.$$  

From (2.23) and (3.13), we see that

$$h_i^*(Y, V_j) = 0.$$  

Taking $X = \xi_i$ to (3.13) and using (2.16) and (2.22), we obtain

$$\ell\theta(V_i) = 0.$$  

Taking the scalar product with $V_j$ and $W_a$ to (2.29) with $\nabla_X V_i = 0$ by turns and using (3.15): $\ell\theta(V_i) = 0$, we have

$$h_i^*(X, \xi_j) = 0, \quad \lambda_{ai}(X) = h_i^*(X, \xi_i) = 0.$$  

Thus, due to (2.32), we see that $M$ is irrotational. \qed

**4 Indefinite complex space forms**

**Definition 4.1.** An indefinite complex space form $\mathcal{M}(c)$ is a connected indefinite Kaehler manifold of constant holomorphic sectional curvature $c$ such that

$$\bar{R}(\bar{X}, \bar{Y})\bar{Z} = \frac{c}{4}\{b(\bar{Y}, \bar{Z})\bar{X} - \bar{g}(\bar{X}, \bar{Z})\bar{Y}$$

$$+ \bar{g}(\bar{J}Y, \bar{Z})\bar{X} - \bar{g}(\bar{J}X, \bar{Z})\bar{Y} + 2\bar{g}(\bar{X}, \bar{J}Y)\bar{Z}\},$$

where $\bar{R}$ is the curvature tensor of the Levi-Civita connection $\bar{\nabla}$ on $\mathcal{M}$. 


Denote by $R$ the curvature tensors of the $(\ell, m)$-type connection $\nabla$ on $M$. By directed calculations from (1.2) and (1.3), we see that

\begin{equation}
R(X, Y)Z = \tilde{R}(X, Y)Z
+ (\nabla_X \theta)(Z)(\ell Y + m J Y)
- (\nabla_Y \theta)(Z)(\ell X + m J X)
+ \theta(Z)((X \ell) Y - (Y \ell) X + (X m) J Y - (Y m) J X).
\end{equation}

Denote by $R$ and $R^*$ the curvature tensor of the induced connections $\nabla$ and $\nabla^*$ on $M$ and $S(TM)$, respectively. Using the Gauss-Weingarten formulae, we obtain Gauss equations for $M$ and $S(TM)$, respectively:

\begin{equation}
\tilde{R}(X, Y)Z = R(X, Y)Z
+ \sum_{i=1}^{r} \{h_i^r(X, Z) A_{\xi_i} Y - h_i^r(Y, Z) A_{\xi_i} X\}
+ \sum_{a=r+1}^{n} \{h_a^s(X, Z) A_{\omega_a} Y - h_a^s(Y, Z) A_{\omega_a} X\}
+ \sum_{i=1}^{r} \{((\nabla_X h_i^r)(Y, Z) - (\nabla_Y h_i^r)(X, Z)
+ \sum_{j=1}^{r} [\tau_{ji}(X) h_{ij}^r(Y, Z) - \tau_{ji}(Y) h_{ij}^r(X, Z)]
+ \sum_{a=r+1}^{n} [\lambda_{ai}(X) h_a^s(Y, Z) - \lambda_{ai}(Y) h_a^s(X, Z)]
- \ell[\theta(X) h_a^s(Y, Z) - \theta(Y) h_a^s(X, Z)]
- m[\theta(X) h_a^s(FY, Z) - \theta(Y) h_a^s(FX, Z)]\} N_i
+ \sum_{a=r+1}^{n} \{((\nabla_X h_a^s)(Y, Z) - (\nabla_Y h_a^s)(X, Z)
+ \sum_{i=1}^{r} [\rho_{ai}(X) h_a^s(Y, Z) - \rho_{ai}(Y) h_a^s(X, Z)]
+ \sum_{b=r+1}^{n} [\sigma_{ba}(X) h_b^s(Y, Z) - \sigma_{ba}(Y) h_b^s(X, Z)]
- \ell[\theta(X) h_a^s(Y, Z) - \theta(Y) h_a^s(X, Z)]
- m[\theta(X) h_a^s(FY, Z) - \theta(Y) h_a^s(FX, Z)]\} E_a,
\end{equation}
(4.4) \[ R(X, Y)PZ = R^*(X, Y)PZ \]
+ \( \sum_{i=1}^{r} \{ h^*_i(X, PZ)A^*_iY - h^*_i(Y, PZ)A^*_iX \} \)
+ \( \sum_{i=1}^{r} \{ (\nabla_X h^*_i)(Y, PZ) - (\nabla_Y h^*_i)(X, PZ) \} \)
+ \( \sum_{k=1}^{r} \{ \tau_{ik}(Y)h^*_k(X, PZ) - \tau_{ik}(X)h^*_k(Y, PZ) \} \)
+ \( \tau_{ik}(Y)h^*_k(Y, PZ) - \theta(Y)h^*_k(X, PZ) \]
+ \( m[\theta(X)h^*_i(FY, PZ) - \theta(Y)h^*_i(FX, PZ)] \}
+ \xi_i \).

Taking the scalar product with \( N_i \) to (4.2) and using (4.1), (4.3) and (4.4), we obtain

(4.5) \[ (\nabla_X h^*_i)(Y, PZ) - (\nabla_Y h^*_i)(X, PZ) \]
- \( \sum_{k=1}^{r} \{ \tau_{ik}(X)h^*_k(Y, PZ) - \tau_{ik}(Y)h^*_k(X, PZ) \} \)
- \( \sum_{k=1}^{r} \{ h^*_k(Y, PZ)\eta_i(A_{n_k}X) - h^*_k(X, PZ)\eta_i(A_{n_k}Y) \} \)
- \( \sum_{u=r+1}^{n} \{ h^*_u(Y, PZ)\eta_i(A_{n_u}X) - h^*_u(X, PZ)\eta_i(A_{n_u}Y) \} \)
- \( \ell[\theta(X)h^*_i(Y, PZ) - \theta(Y)h^*_i(X, PZ)] \)
- \( m[\theta(X)h^*_i(FY, PZ) - \theta(Y)h^*_i(FX, PZ)] \)
- \( (\nabla_X \theta)(PZ) \{ \ell\eta_i(Y) + mv_i(Y) \} + (\nabla_Y \theta)(PZ) \{ \ell\eta_i(X) + mv_i(X) \} \)
- \( \theta(PZ) \{ (X\ell)\eta_i(Y) - (Y\ell)\eta_i(X) + (Xm)v_i(Y) - (Ym)v_i(X) \} \)
+ \( \frac{c}{4} \{ g(Y, PZ)\eta_i(X) - g(X, PZ)\eta_i(Y) \}
+ v_i(Y)g(Y, PZ) - v_i(Y)g(JX, PZ) + 2v_i(PZ)g(X, JY) \}.

**Theorem 4.1.** Let \( M \) be a generic lightlike submanifold of an indefinite complex space form \( M(c) \) with an \((\ell, m)\)-type connection such that \( \xi \) belongs to \( S(TM) \). If either \( U_i \)s or \( V_i \)s are parallel with respect to the connection \( \nabla \), then \( c = 0 \) and \( M(c) \) is flat.

**Proof.** (1) Assume that \( U_i \)s are parallel with respect to the connection \( \nabla \). Applying \( \nabla_X \) to (3.11) and using the fact that \( \nabla_X U_j = 0 \), we get

(4.6) \[ (X\ell)\theta(U_j) + \ell(\nabla_X \theta)(U_j) = 0, \quad (Xm)\theta(U_j) + m(\nabla_X \theta)(U_j) = 0. \]

Applying \( \nabla_X \) to (3.12) and using the fact that \( \nabla_X U_j = 0 \), we get

(4.7) \[ (\nabla_X h^*_i)(Y, U_j) = 0. \]

Replacing \( Z \) by \( U_j \) to (4.5) and using (3.11), (3.12), (4.6) and (4.7), we obtain

\[ \frac{c}{4} \{ v_j(Y)\eta_i(X) - v_j(X)\eta_i(Y) - v_i(X)\eta_j(Y) + v_i(Y)\eta_j(X) \} = 0. \]
Taking $Y = V_j$ and $X = \xi_i$ to this equation, we obtain $c = 0$.

(2) Assume that $V_\ell$s are parallel with respect to the connection $\nabla$ of $M$. Applying $\nabla_X$ to (3.14) and using the fact that $\nabla_X V_j = 0$, we get

$$\nabla h^i_j(Y, V_j) = 0.$$ 

Taking $X = V_k$ to (3.13), we obtain

$$h^i_k(V_k, U_j) = -m\theta(V_i)\delta_{jk}.$$ 

Taking $X = V_k$ and $Y = U_j$ to (2.16) and using (4.9), we get

$$h^j_i(U_j, V_k) = m\{\theta(V_k)\delta_{ij} - \theta(V_i)\delta_{jk}\}. $$

Taking $X = U_k$ to (2.26) and using (4.10), we have

$$h^j_k(U_k, V_i) = 0.$$ 

Taking $i = j$ to (4.10) and using (4.11) and the fact that $r > 1$, we obtain

$$m\theta(V_k) = 0.$$ 

Applying $\nabla_X$ to (3.15): $\ell\theta(V_i) = 0$ and (4.12): $m\theta(V_i) = 0$ by turns and using the fact that $\nabla_X V_j = 0$, we get

$$(X\ell)\theta(V_j) + \ell(\nabla_X \theta)(V_j) = 0, \quad (Xm)\theta(V_j) + m(\nabla_X \theta)(V_j) = 0.$$ 

Replacing $X$ by $W_a$ to (3.13), we obtain

$$h^i_j(W_a, U_j) = 0.$$ 

Taking $X = W_a$ and $Y = U_j$ to (2.16) and using the last equation, we get

$$h^j_i(U_j, W_a) = m\theta(W_a)\delta_{ij}.$$ 

Replacing $X = U_j$ to (2.25) and using the last equation, we have

$$h^i_a(U_j, V_i) = 0.$$ 

Taking $Z = V_j$ to (4.5) and using (3.14), (4.8) and (4.13), we obtain

$$- c \sum_{k=1}^r \{h^i_k(Y, V_j)\eta_i(A_{Nk} X) - h^i_k(X, V_j)\eta_i(A_{Nk} Y)\}$$

$$- \sum_{a=r+1}^n \{h^i_a(Y, V_j)\eta_i(A_{ea} X) - h^i_a(X, V_j)\eta_i(A_{ea} Y)\}$$

$$= \frac{c}{4} \{u_j(Y)\eta_i(X) - u_j(X)\eta_i(Y) + 2\delta_{ij}\tilde{g}(X, JY)\}.$$ 

Taking $Y = U_j$ and $X = \xi_i$ to this equation and using (2.16), (2.17), (3.16), (4.11) and (4.14), we obtain $c = 0$. 

$\square$
Theorem 4.2. Let $M$ be a solenoidal generic lightlike submanifold of an indefinite complex space form $\tilde{M}(c)$ with an $(\ell, m)$-type connection such that $\zeta$ belongs to $S(TM)$. If $W_a$s are parallel with respect to $\nabla$, then $c = 0$.

Proof. Assume that $W_a$s are parallel with respect to $\nabla$. Taking the scalar product with $W_b$ to (2.30) with $\nabla X W_a = 0$, we have
\[(4.15) \quad \mu_{ab}(X) = -\ell \theta(W_a) w_b(X).\]
From (4.15) and the fact that $\epsilon_b \mu_{ab} + \epsilon_a \mu_{ba} = 0$, we have
\[\ell \{ \epsilon_b \theta(W_a) w_b(X) + \epsilon_a \theta(W_b) w_a(X) \} = 0.\]
Taking $X = \epsilon_a W_b$ to this equation, we obtain
\[(4.16) \quad \ell \theta(W_a) = 0, \quad \mu_{ab} = 0.\]

Taking the scalar product with $V_i$, $U_i$ and $N_i$ to (2.30) with $\nabla X W_a = 0$ by turns and using (2.19) and (4.16), we have
\[(4.17) \quad \lambda_{ai} = 0, \quad \eta_i(A_{x_a} X) = -m \theta(W_a) \eta_i(X), \quad h_{ai}^*(X, U_i) = m \{ \theta(U_i) w_a(X) - \epsilon_a \theta(W_a) \eta_i(X) \}.\]
From (2.24) and (4.17), we obtain
\[(4.18) \quad h_{ai}^*(X, W_a) = 0.\]
Applying $\nabla X$ to (4.18) and using the fact that $\nabla X W_a = 0$, we get
\[(4.19) \quad (\nabla X h_{ai}^*)(Y, W_a) = 0.\]
Now we shall assume that $M$ is solenoidal. Then we obtain (2.33):
\[(4.20) \quad \eta_i(A_{x_j} X) = 0, \quad \eta_i(A_{x_a} X) = 0.\]
from the second equation of the last equations and (4.17), we obtain
\[m \theta(W_a) = 0.\]
Applying $\nabla X$ to $\ell \theta(W_a) = 0$ and $m \theta(W_a) = 0$ and using the fact that $\nabla X W_a = 0$, we get
\[(4.21) \quad (X \ell) \theta(W_a) + \ell(\nabla X \theta)(W_a) = 0, \quad (X m) \theta(W_a) + m(\nabla X \theta)(W_a) = 0.\]
Taking $X = W_a$ to (4.5) and using (4.18)~(4.21), we have
\[c \{ w_a(Y) \eta_i(X) - w_a(X) \eta_i(Y) \} = 0.\]
Taking $Y = W_a$ and $X = \xi_i$ to this equation, we obtain $c = 0$. \hfill \square

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