THE GENERALIZED HANKEL-CLIFFORD TRANSFORMATION WITH COMPACT SUPPORT ON CERTAIN RANGE

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Abstract. The Paley-Wiener theorem for the generalized Hankel-Clifford transforms is obtained. The generalized Hankel-Clifford transforms of square integrable functions with compact supports, rapid decreasing functions, infinitely differentiable functions with compact supports, of analytic functions are studied. The range of the generalized Hankel-Clifford transform of compactly supported functions which are either square integrable (Paley-Wiener Theorem) or infinitely differentiable (Paley-Wiener-Schwartz Theorem) is characterized. Such developed transforms are supported by an application to Mathematical Physics at the end of the section of the study.

1. Introduction

The generalized Hankel-Clifford transformations defined by

\[ f(x) = (h_{1,\alpha,\beta}g)(x) = x^{-(\alpha+\beta)} \int_{0}^{\infty} J_{\alpha,\beta}(xy) g(y) \, dy, \quad (1.1) \]

and

\[ p(x) = (h_{2,\alpha,\beta}t)(x) = \int_{0}^{\infty} y^{-(\alpha+\beta)} J_{\alpha,\beta}(xy) t(y) \, dy \quad (1.2) \]

if the integral converges in some sense (absolutely, improper, or mean convergence). Here \( J_{\alpha,\beta}(z) = z^{(\alpha+\beta)/2} J_{\alpha-\beta}(2\sqrt{z}) \), \( J_{\alpha-\beta}(z) \) being the Bessel function of the first kind and order \( (\alpha - \beta) \geq -1/2 \) were extended by Malgonde [1] to certain generalized functions [6]. It is analogous from [5] and as represented in [2] that if \( Re(\alpha - \beta) \geq -1/2 \), then the generalized Hankel-Clifford transformations is an automorphism of \( L_2(R_+) \) and the inverse generalized Hankel-Clifford transformations on \( L_2(R_+) \) has the symmetric form

\[ g(x) = (h_{1,\alpha,\beta}f)(x) = x^{-(\alpha+\beta)} \int_{0}^{\infty} J_{\alpha,\beta}(xy) f(y) \, dy, \quad (1.3) \]
\[ t(x) = (h_{2,\alpha,\beta}p)(x) = \int_{0}^{\infty} y^{-(\alpha+\beta)} J_{\alpha,\beta}(xy) p(y) \, dy. \]  

(1.4)

Let us take note here of some properties of Bessel functions that we shall use quite a few times in this work (see [4]).

**Definition 1.1.** The behaviors of \( J_{\alpha-\beta} \) near the origin and the infinity are from [8] as follows:

\[ J_{\alpha-\beta}(2x^{1/2}) = O\left(x^{1/2}\right)^{\alpha-\beta} \]  

(1.5)
as \( x \to 0+\).

\[ J_{\alpha-\beta}\left(2x^{1/2}\right) \approx (2\pi)x^{-1/4}\cos\left(2x^{1/2} - \frac{1}{2}(\alpha - \beta)\pi - \frac{1}{4}\pi\right) \sum_{m=0}^{\infty} \frac{(-1)^{m}(\alpha - \beta, 2m)}{(4x^{1/2})^{2m}} \]

\[ -\sin\left(2x^{1/2} - \frac{1}{2}(\alpha - \beta)\pi - \frac{1}{4}\pi\right) \sum_{m=0}^{\infty} \frac{(-1)^{m}(\alpha - \beta, 2m + 1)}{(4x^{1/2})^{2m+1}} \]  

(1.6)
as, \( x \to \infty \) where \( (\alpha - \beta, k) \) is understood as in [4].

**Definition 1.2.** The main differentiation formulas for \( J_{\alpha-\beta} \) in [1] are:

\[ \frac{d}{dx}\left[x^{(\alpha-\beta)/2} J_{\alpha-\beta}(2\sqrt{x})\right] = x^{(\alpha-\beta-1)/2} J_{\alpha-\beta-1}(2\sqrt{x}) . \]  

(1.7)

\[ \frac{d}{dx}\left[x^{-(\alpha-\beta)/2} J_{\alpha-\beta}(2\sqrt{x})\right] = -x^{-(\alpha-\beta+1)/2} J_{\alpha-\beta+1}(2\sqrt{x}) . \]  

(1.8)

\[ x^{\alpha+\beta+1} \frac{d}{dx}\left[x^{-(\alpha-\beta)/2} J_{\alpha-\beta}(2\sqrt{x})\right] = -x^{\alpha+\beta+1/2} J_{\alpha-\beta+1}(2\sqrt{x}) \]  

(1.9)

for \( x, y > 0 \).

**Definition 1.3.** The generalized Kepinski type differential operator from [1] is defined as

\[ \Delta_{\alpha,\beta} = \Delta_{\alpha,\beta,x} = x^{-\alpha}Dx^{\alpha+\beta+1}x^{\beta} =xD^{2} + (\alpha - \beta + 1)D + \alpha x^{-1} \]  

(1.10)

where \( \alpha - \beta \geq -1/2 \) and \( D = \frac{d}{dx} \).

**Property 1.1.** By combining (1.7) and (1.8) and (1.10), it can be easily inferred

\[ \Delta_{\alpha,\beta} J_{\alpha,\beta}(x) = -J_{\alpha,\beta}(x) \]  

(1.11)

**Property 1.2.** The generalized Hankel-Clifford transforms can be extended to

\[ f(x) = (h_{1,\alpha,\beta}g)(x) = x^{-(\alpha+\beta)} \int_{0}^{\infty} J_{\alpha,\beta,m}(xy) g(y) \, dy, \]  

(1.12)

where \( J_{\alpha,\beta,m}(x) = x^{(\alpha+\beta)/2} J_{\alpha-\beta,m}(2\sqrt{x}) \) and \( J_{\alpha-\beta,m}(2\sqrt{x}) \) being the truncated Bessel function of the first kind analogous to [2] and is represented as

\[ J_{\alpha-\beta,m}(2\sqrt{x}) = J_{\alpha-\beta}(2\sqrt{x}) - \sum_{k=0}^{m-1} \frac{(-1)^{k}(\sqrt{x})^{(\alpha-\beta+2k)}}{\Gamma(\alpha - \beta + k + 1)k!} \]
and the integral is taken in sense of $L_2$.

The generalized Hankel-Clifford transforms and its inverse will have a bounded operator in $L_2(R_+)$ from [9] and has been extended from [2] as:

$$g(x) = x^{-(3\alpha + \beta - 1)/2} \frac{d}{dx} x^{(3\alpha - \beta + 1)/2} \int_0^\infty x^{(\alpha - \beta + 1)} J_{\alpha - \beta + 1, m+1} (2\sqrt{xy}) f(y) \, dy$$

for $x \in R_+; 1/2 - m < \text{Re} (\alpha - \beta) < m + 1/2$, $m > 0$.

$$J_{\alpha - \beta - 1, m+1} (x) = x^{-1/2} J_{\alpha - \beta + 1, m+1} (2\sqrt{x}).$$

Property 1.3. Using the equivalent form of the [equation (7); 2], we get

$$\frac{d}{dx} \left[ x^{(\alpha - \beta + 1)} J_{\alpha - \beta - 1, m+1} (2\sqrt{x}) \right] = x^{(\alpha - \beta) + 1/2} J_{\alpha - \beta, m} (2\sqrt{x}),$$

where $\text{Re} (\alpha - \beta) < m + 1/2, m > 0$.

Then symmetric to formula [8; 2] can be extended to

$$g_N(x) = x^{-(3\alpha + \beta - 1)/2} \frac{d}{dx} x^{(3\alpha - \beta + 1)/2} \int_{1/N}^N x^{(\alpha - \beta + 1)} J_{\alpha - \beta + 1, m+1} (2\sqrt{xy}) f(y) \, dy$$

$$= x^{-(\alpha + \beta)} \int_{1/N}^N J_{\alpha, m} (xy) g(y) \, dy$$

In this paper, the range of the generalized Hankel-Clifford transforms on some spaces of functions has been described. The range of the generalized Hankel-Clifford transforms of compactly supported functions which are either square integrable (Paley-Wiener Theorem) or infinitely differentiable (Paley-Wiener-Schwartz Theorem) is also characterized.

One of the main tools of our next two theorems is the Plancherel’s theorem for the generalized Hankel-Clifford transformations as proved in [10] can be represented as

$$\|h_{1,\alpha,\beta}g\|_2 = \|g\|_2$$

where $\|g\|_p = \|g\|_{L_p(R_+)}$, $1 \leq p < \infty$, that is valid only when $(\alpha - \beta) \geq -1/2$.

For complex $(\alpha - \beta)$, the Plancherel’s equation is replaced by the inequalities

$$C^{-1}\|g\|_2 \leq \|h_{1,\alpha,\beta}g\|_2 \leq C\|g\|_2, (\alpha - \beta) \geq -1/2$$

where $C \in [1, \infty)$ is a constant independent of $g$.

2. RANGE OF THE GENERALIZED HANKEL-CLIFFORD TRANSFORMS OF RAPID DECREASING AND SQUARE INTEGRABLE FUNCTIONS

The range of the generalized Hankel-Clifford transforms of rapid decreasing and square integrable functions is described by the following:

Theorem 2.1. Let $y^m g(y) \in L_2(R_+)$ for all $m = 0, 1, 2, 3$. A function $f(x)$ be the generalized Hankel-Clifford transform $h_{1,\alpha,\beta}$ of $g(y)$ order $\text{Re} (\alpha - \beta) \geq -1/2$ if and only if:

i) $f(x)$ is infinitely differentiable on $R_+$.

ii) $\Delta_{\alpha, \beta, x}^m f(x), m = 0, 1, 2, 3, \ldots, belong to $L_2(R_+)$;

iii) $\Delta_{\alpha, \beta, x}^m f(x), m = 0, 1, 2, 3, \ldots, tends to 0 as x tends to 0 and to infinity;
\( \Delta_{\alpha, \beta}^{m, k} f(x) \), \( m = 0, 1, 2, 3... \) tends to 0 as \( x \) tends to infinity and are bounded at 0.

Proof. Necessary:

i) Let \( y^{m}g(y) \in L_{2}(R_{+}) \) for all \( m = 0, 1, 2, 3... \) then \( y^{m}g(y) \in L_{1}(R_{+}) \) for all \( m = 0, 1, 2, 3... \)

Let \( f(x) \) be the generalized Hankel-Clifford transform \( h_{1, \alpha, \beta} \) of \( g(y) \). Indeed, it is easily verified that (\([2, 4]\)).

\[
\frac{\partial^{m}}{\partial y^{m}} \left( y^{-\alpha-\beta}(xy)^{(\alpha+\beta)/2} J_{\alpha-\beta} (2\sqrt{xy}) \right) = \sum_{j=0}^{m} a_{j}(\alpha) y^{-\left(\frac{\alpha+\beta+1}{2}\right)} y^{j-m} x^{\left(\frac{\alpha+\beta+1}{2}\right)} J_{\alpha-\beta-j} (2\sqrt{xy})
\]

(2.1)

where the \( a_{j}(\alpha) \) are constants depending on \( \alpha \) only.

Considering

\[
D^{k} \left[ x^{-\alpha}(xy)^{\left(\frac{\alpha+\beta}{2}\right)+j/2} J_{\alpha-\beta-j} (2\sqrt{xy}) \right] = y^{\alpha} D^{k} \left[ (xy)^{-\left(\frac{\alpha+\beta}{2}\right)+j/2} J_{\alpha-\beta-j} (2\sqrt{xy}) \right]
\]

\[
= (-1)^{k} y^{(\alpha+\beta+k)/2} \left[ x^{-\left(\alpha-\beta-j+k\right)/2} J_{\alpha-\beta-j+k} (2\sqrt{xy}) \right]
\]

\[
= O \left[ (xy)^{-\left(\frac{\alpha-\beta-j+k}{2}\right)-1/4} e^{\sqrt{\pi}} |\Im \sqrt{\pi}| \right]
\]

as \( x \to 0^{+} \)

It follows that

\[
\gamma_{m,k}^{\alpha, \alpha} \left( y^{(-\alpha-\beta)/2} x^{(\alpha-\beta-j)/2} J_{\alpha-\beta-j} (2\sqrt{xy}) \right) < \infty.
\]

Therefore

\[
\gamma_{m,k}^{\alpha, \alpha} \left[ \frac{\partial^{m}}{\partial y^{m}} \left( y^{(-\alpha-\beta)} (xy)^{(\alpha+\beta)/2} J_{\alpha-\beta} (2\sqrt{xy}) \right) \right]
\]

\[
\leq \sum_{j=0}^{m} |a_{j}(\alpha)| y^{\frac{1}{2}-m} \gamma_{m,k}^{\alpha, \alpha} \left( y^{(-\alpha-\beta)/2} x^{(\alpha-\beta-j)/2} J_{\alpha-\beta-j} (2\sqrt{xy}) \right) < \infty.
\]

for a fixed \( y \in \Omega \).

ii) Since \( x^{\left(\frac{\alpha+\beta}{2}\right)} J_{\alpha-\beta} (2\sqrt{xy}) \) is the solution of differential equation by Malgonde and Lakshmi Gorty in [8]

\[
f''(x) + (1 - \alpha - \beta) x^{-1} f'(x) + (\alpha \beta x^{-2} + 1) f(x) = 0.
\]

(2.2)
Therefore
\[ \Delta_{\alpha,\beta,x}^m \{ x^{-\alpha-\beta} J_{\alpha,\beta}(xy) \} = (-y)^m \{ x^{-\alpha-\beta} J_{\alpha,\beta}(xy) \}. \] (2.3)

Consequently
\[ \Delta_{\alpha,\beta,x}^m \{ x^{-\alpha-\beta} J_{\alpha,\beta}(xy) \} = (-1)^m \int_0^\infty x^{-\alpha-\beta} J_{\alpha,\beta}(xy) y^m g(y) dy, \] (2.4)

with \((\alpha - \beta) > -1/2\). Plancherel’s inequality gives \(y^m g(y) \in L_2(\mathbb{R}_+)\), and \(\Delta_{\alpha,\beta,x}^m \{ x^{-\alpha-\beta} J_{\alpha,\beta}(xy) \} \), \((\alpha - \beta) \geq -1/2\), \(m = 0, 1, 2, 3, \ldots \in L_2(\mathbb{R}_+)\).

iii) For the kernel \(x^{-\alpha-\beta} J_{\alpha,\beta}(xy)\) has asymptotes \(x^{(\alpha-\beta+1)/2}\) as \(x\) tends to 0, is uniformly bounded on \((0, \infty)\) if \((\alpha - \beta) \geq -1/2\) and \(y^m g(y) \in L_1(0, \infty)\), then applying dominated convergence theorem,
\[ \lim_{x \to \infty} \{ \Delta_{\alpha,\beta,x}^m \{ f(x) \} \} = (-1)^m \int_0^\infty x^{-\alpha-\beta} J_{\alpha,\beta}(xy) y^m g(y) dy = 0. \] (2.5)

\((\alpha - \beta) \geq -1/2\).

For every \(\varepsilon > 0\) one can choose large enough so that
\[ \left| \int_N^\infty x^{-\alpha-\beta} J_{\alpha,\beta}(xy) y^m g(y) dy \right| < \varepsilon. \] (2.6)
uniformly with respect to \(x \in \mathbb{R}_+\).

By applying the generalized Riemann-Lebesgue theorem,
\[ \lim_{x \to \infty} \int_0^N x^{-\alpha-\beta} J_{\alpha,\beta}(xy) y^m g(y) dy = 0, \] (2.7)
\(0 \leq N < \infty, (\alpha - \beta) \geq -1/2\).

Because \(\varepsilon\) can be taken arbitrarily small,
\[ \lim_{x \to \infty} \int_0^\infty x^{-\alpha-\beta} J_{\alpha,\beta}(xy) y^m g(y) dy = 0, \] (2.8)
\(0 \leq N \leq \infty, (\alpha - \beta) \geq -1/2\).

Hence
\[ \lim_{x \to \infty} \{ \Delta_{\alpha,\beta,x}^m \{ f(x) \} \} = 0, \ m = 0, 1, 2, \ldots, (\alpha - \beta) \geq -1/2. \] (2.9)

iv) Using (1.6), we get
\[ (-1)^m \frac{d}{dx} \{ \Delta_{\alpha,\beta,x}^m f(x) \} = \int_0^\infty x^{(\alpha - \beta - 1)/2} J_{\alpha-\beta-1} (2\sqrt{x}) y^m g(y) dy. \] (2.10)

From (2.5) and (2.8) of (iii), we can state that the right hand side of (2.10) tends to zero as \(x\) tends to infinity. Since \(x^{(\alpha - \beta - 1)/2} J_{\alpha-\beta-1} \) is uniformly bounded on \(\mathbb{R}_+\), therefore the right hand side of (2.10) is also uniformly bounded.

Sufficiency:
If \(f(x)\) satisfies the conditions i) to iv) of the theorem 2.1.
Then $\Delta_{\alpha,\beta}^m f(x)$, $m = 0, 1, 2, 3, \ldots$, belong to $L_2(R_+)$.

Let $g_m(y)$ be its generalized Hankel-Clifford transform:

$$g_m(y) = \int_0^\infty x^{-\alpha-\beta} \mathcal{J}_{\alpha,\beta}(xy) \Delta_{\alpha,\beta}^m f(x) \, dx; \quad m = 0, 1, 2, 3, \ldots \quad (2.11)$$

$(\alpha - \beta) \geq -1/2$.

Since

$$g_m^N(y) = \int_{1/N}^N x^{-\alpha-\beta} \mathcal{J}_{\alpha,\beta}(xy) \Delta_{\alpha,\beta}^m f(x) \, dx; \quad m = 0, 1, 2, 3, \ldots \quad (2.12)$$

$(\alpha - \beta) \geq -1/2$.

Here $g_m^N(y) \to g_m(y)$ in $L_2$ norm as $N \to \infty$.

Integrating (2.12) by parts twice,

$$g_m^N(y) = x^{-\alpha-\beta} \mathcal{J}_{\alpha,\beta}(xy) \frac{d}{dx} \Delta_{\alpha,\beta}^{m-1} f(x) \bigg|_{1/N}^N - \frac{\partial}{\partial x} \{x^{-\alpha-\beta} \mathcal{J}_{\alpha,\beta}(xy)\} \Delta_{\alpha,\beta}^{m-1} f(x) \bigg|_{1/N}^N + \int_{1/N}^N \Delta_{\alpha,\beta} \{x^{-\alpha-\beta} \mathcal{J}_{\alpha,\beta}(xy)\} \Delta_{\alpha,\beta}^{m-1} f(x) \, dx. \quad (2.13)$$

The following can be concluded:

a) $x^{-\alpha-\beta} \mathcal{J}_{\alpha,\beta}(xy)$ is uniformly bounded and $\frac{d}{dx} \Delta_{\alpha,\beta}^{m-1} f(N) \to 0$ as $N \to \infty$.

b) $\frac{d}{dx} \Delta_{\alpha,\beta}^{m-1} f(N^{-1})$ is bounded, whereas $N^{-\alpha-\beta-1} \mathcal{J}_{\alpha,\beta}(N^{-1}y)$ has an order $O\left(N^{-\alpha-\beta-1/2}\right)$ as $N \to \infty$.

c) $(\alpha + \beta) N^{-\alpha-\beta-1} \{N^{-\alpha-\beta} \mathcal{J}_{\alpha,\beta}(Ny)\}$ and $\Delta_{\alpha,\beta}^{m-1} f(N)$ is of $O(1)$.

d) $N^{-\alpha-\beta} N^{-\alpha-\beta-1} \{\mathcal{J}_{\alpha,\beta-1}(N)\}$ and $\Delta_{\alpha,\beta}^{m-1} f(N)$ is of $O(1)$, tends to zero as $N \to \infty$.

e) $(\alpha + \beta) N^{-\alpha-\beta-2} \{N^{-\alpha-\beta-1} \mathcal{J}_{\alpha,\beta}(N^{-1}y)\}$ and $\Delta_{\alpha,\beta}^{m-1} f(N^{-1})$ is of $O(1)$, tends to zero as $N \to \infty$.

f) $\int_{1/N}^N \Delta_{\alpha,\beta} \{x^{-\alpha-\beta} \mathcal{J}_{\alpha,\beta}(xy)\} \Delta_{\alpha,\beta}^{m-1} f(x) \, dx$ converges to $(-y) g_{m-1}(y)$ as $N \to \infty$. 

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Hence \( g_m (y) = (-y) g_{m-1} (y) \), therefore \( g_m (y) = (-y)^m g_0 (y) \), \( m = 0, 1, 2, \ldots \).

But \( f \) is the generalized Hankel-Clifford transforms of \( g \). Thus \( f(x) \) is the generalized Hankel-Clifford transforms of the function \( g(y) = g_0 (y) \) such that \( y^m g(y) \in L_2 (R_+) \), \( n = 0, 1, 2, \ldots \) and theorem 2.1 is thus proved.

3. Generalized Hankel-Clifford transform of infinitely differentiable functions with compact supports

**Theorem 3.1.** *(Paley-Wiener theorem for the generalized Hankel-Clifford transforms of square integrable functions with compact supports)* A function \( f \) is the generalized Hankel-Clifford transforms of a square integrable function \( g \) with compact support on \([0, \infty)\) if and only if \( f \) satisfies conditions i)-iv) of Theorem 2.1 and

\[
\lim_{n \to \infty} \left\| \Delta_{\alpha, \beta, x}^m f(x) \right\|_2^{1/2m} = \sigma_g < \infty,
\]

where \( \sigma_g = \sup \{ y : y \in \text{supp} \ g \} \) and the support of a function is the smallest closed set, outside it the function vanishes almost everywhere.

Proof. Necessary: Let \( f(x) \) be the generalized Hankel-Clifford transforms of \( g(y) \in L_2 (R_+) \) and assuming \( \sigma_g > 0 \) and \( \sigma_g < \infty \):

\[
f(x) = x^{-(\alpha + \beta)} \int_0^{\sigma_g} \mathcal{J}_{\alpha, \beta} (xy) g(y) \, dy
\]

\( y^m g(y) \in L_2 (R_+) \), \( \forall m = 0, 1, 2, \ldots \), \( f \) satisfies conditions i)-iv) of theorem 2.1.

Invoking the right side of the inequality \[(1.17)\] in \[(3.2)\], we get:

\[
\left\| \Delta_{\alpha, \beta, x}^m f(x) \right\|_2^2 \leq C \int_0^{\sigma_g} y^{2m} |g(y)|^2 \, dy \leq C \int_0^{\sigma_g} \sigma_g^{2m} |g(y)|^2 \, dy.
\]

Hence

\[
\lim_{m \to \infty} \left\| \Delta_{\alpha, \beta, x}^m f(x) \right\|_2 = \lim_{m \to \infty} C^{1/2m} \sigma_g \left\{ \int_0^{\sigma_g} |g(y)|^2 \, dy \right\}^{1/2m} = \sigma_g.
\]

Since \( \sigma_g \) is the least upper bound of the support of \( g \), for every \( \varepsilon, 0 < \varepsilon < \sigma_g \), gives \( \int_{\sigma_g - \varepsilon}^{\sigma_g} |g(y)|^2 \, dy > 0 \).

Consequently left side of the inequality in \[(1.17)\], gives

\[
\lim_{m \to \infty} \left\| \Delta_{\alpha, \beta, x}^m f(x) \right\|_2 \geq \lim_{m \to \infty} C^{-1/2m} (\sigma_g - \varepsilon) \left\{ \int_{\sigma_g - \varepsilon}^{\sigma_g} |g(y)|^2 \, dy \right\}^{1/2m} = \sigma_g - \varepsilon.
\]

Sufficient:

Suppose now that \( f \) satisfies the conditions i)-iv) of theorem 2.1, and the limit in \[(3.1)\] exists and equals \( \sigma < \infty \).

Using theorem 2.1, \( f \) is the generalized Hankel-Clifford transforms of a function \( g \) such that \( y^m g(y) \in L_2 (R_+) \), \( \forall m = 0, 1, 2, \ldots \) It is to be proved that \( \sigma < \infty \) and
σ = σ_\text{g}. From theorem 2.1 it is observed that \(2.4\) is valid. Therefore, applying the inequalities \([1.17]\) it is obtained as:

\[
C^{-1} \| y^m g(y) \|_2 \leq \| \Delta^m_{\alpha,\beta,\varepsilon} f(x) \|_2 \leq C \| y^m g(y) \|_2.
\]

Hence

\[
\lim_{m \to \infty} C^{-1} \| y^m g(y) \|_2 \leq \lim_{m \to \infty} \| \Delta^m_{\alpha,\beta,\varepsilon} f(x) \|_2 \leq \lim_{m \to \infty} C \| y^m g(y) \|_2 \leq \lim_{m \to \infty} C \| \sigma^m g(y) \|_2.
\]

Consequently

\[
\lim_{m \to \infty} \| y^m g(y) \|_2^{1/2m} = \sigma.
\]

Suppose that σ_\text{g} > \sigma. Then there exists a positive \(\varepsilon\) such that

\[
\int_{\sigma + \varepsilon}^{\infty} |g(y)|^2 dy > 0.
\]

Then

\[
\sigma = \lim_{m \to \infty} \| y^m g(y) \|_2^{1/2m} \geq \lim_{m \to \infty} \left\{ \int_{\sigma + \varepsilon}^{\infty} y^{2m} |g(y)|^2 dy \right\}^{1/2m}
\]

\[
\geq \lim_{m \to \infty} (\sigma + \varepsilon) \left\{ \int_{\sigma + \varepsilon}^{\infty} |g(y)|^2 dy \right\}^{1/2m} = \sigma + \varepsilon.
\]

which is impossible. Hence σ_\text{g} ≤ \sigma and \(g\) has a compact support.

Suppose that σ_\text{g} < \sigma. Then there exists a positive \(\varepsilon\) such that

\[
\int_{0}^{\sigma - \varepsilon} |g(y)|^2 dy > 0.
\]

Thus

\[
\sigma = \lim_{m \to \infty} \| y^m g(y) \|_2^{1/2m} \leq \lim_{m \to \infty} \left\{ \int_{0}^{\sigma - \varepsilon} y^{2m} |g(y)|^2 dy \right\}^{1/2m}
\]

\[
\leq \lim_{m \to \infty} (\sigma - \varepsilon) \left\{ \int_{0}^{\sigma - \varepsilon} |g(y)|^2 dy \right\}^{1/2m} = \sigma - \varepsilon.
\]

which is impossible. Hence σ_\text{g} ≥ \sigma and thus σ = σ_\text{g}. Thus the theorem is proved.

4. Generalized Erdelyi-Kober Fractional Integral Operator

Let the generalized Erdelyi-Kober fractional integral operator as defined by [7]

\[
h(x) = (K_{\alpha,\beta}g_1)(x) = \frac{2(\alpha+\beta)/2}{\Gamma\left(\frac{\alpha-\beta+1}{2}\right)} \int_{x}^{\infty} (y^2 - x^2)^{\left(\frac{\alpha+\beta-1}{2}\right)} y g_1(y) dy;
\]

where Re(\(\alpha - \beta\)) > 0; \(x \in R\).
Theorem 4.1. (Paley-Wiener-Schwartz theorem for generalized Hankel-Clifford transform of infinitely differentiable functions with compact supports) A function \( f \in h_{1,\alpha,\beta} \) is the generalized Hankel-Clifford transform for \((\alpha - \beta) \geq -1/2\) of a function \( g \in h_{1,\alpha,\beta} \) with compact support if and only if

\[
\lim_{m \to \infty} \left\| \frac{d^m}{dx^m} x^\alpha f (x) \right\|_p^{1/m} = \sigma_g, 1 \leq p \leq \infty. \tag{4.2}
\]

Proof. The integral representation of generalized Hankel-Clifford function \( J_{\alpha,\beta} (xy) \) analogously can be written as \([3]\),

\[
J_{\alpha,\beta} (x) = \frac{2^{1+\alpha+\beta}/2 \, x^{-(\alpha+\beta)/2} y^{(\alpha+\beta)/2}}{\sqrt{\pi \Gamma \left( \frac{\alpha-\beta+1}{2} \right)}} \int_0^1 (1-t^2)^{\left( \frac{\alpha+\beta-1}{2} \right)} \cos (2t \sqrt{x}) \, dt, \tag{4.3}
\]

\( \text{Re} (\alpha - \beta) \geq -1/2. \) Substituting \( x \) by \( xy^2 \) and \( t \) by \( t/y \), it gives

\[
J_{\alpha,\beta} (xy) = \frac{2^{1+\alpha+\beta}/2 \, x^{-(\alpha+\beta)/2} y^{3(\alpha+\beta)/2}}{\sqrt{\pi \Gamma \left( \frac{\alpha-\beta+1}{2} \right)}} \int_0^y (y^2-t^2)^{\left( \frac{\alpha+\beta-1}{2} \right)} y^{-(\alpha+\beta-1)} \cos (2t \sqrt{x}) \, dt. \tag{4.4}
\]

The generalized Hankel-Clifford transform can be rewritten as

\[
f (x) = \frac{2^{1+\alpha+\beta}/2 \, x^{-(\alpha+\beta)/2} y^{(\alpha+\beta)/2}}{\sqrt{\pi \Gamma \left( \frac{\alpha-\beta+1}{2} \right)}} \int_0^\infty y^{-(\alpha+\beta-1)} g (y) \int_0^y (y^2-t^2)^{\left( \frac{\alpha+\beta-1}{2} \right)} \cos (2t \sqrt{x}) \, dt \, dy. \tag{4.5}
\]

If \( y^{-(\alpha+\beta)/2} g (y) \in L_1 (R_+) \), then the repeated integral \((4.5)\) converges absolutely. Therefore, Fubini-Tonelli theorem \([5]\) is applied to interchange the order of integration in \((4.5)\):

\[
f (x) = \frac{2^{1+\alpha+\beta}/2 \, x^{-(\alpha+\beta)/2} y^{(\alpha+\beta)/2}}{\sqrt{\pi \Gamma \left( \frac{\alpha-\beta+1}{2} \right)}} \int_0^\infty \cos (2t \sqrt{x}) \, dt \int_0^\infty (y^2-t^2)^{\left( \frac{\alpha+\beta-1}{2} \right)} y g (y) \, dy. \tag{4.6}
\]

Considering \( f_1 (x) = x^{(\alpha+\beta)/2} f (x) \) and \( g_1 (y) = y^{-(\alpha+\beta)/2} g (y) \),

\[
f_1 (x) = \frac{2^{1+\alpha+\beta}/2}{\sqrt{\pi \Gamma \left( \frac{\alpha-\beta+1}{2} \right)}} \int_0^\infty \cos (2t \sqrt{x}) \, dt \int_0^\infty (y^2-t^2)^{\left( \frac{\alpha+\beta-1}{2} \right)} y g_1 (y) \, dy. \tag{4.7}
\]

Therefore \( f_1 (x) \) can be viewed as composition of the Fourier cosine transform

\[
f_1 (x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \cos (2t \sqrt{x}) \, h (t) \, dt, \quad 0 \leq x < \infty, \tag{4.8}
\]

where

\[
h (t) = \frac{2^{(\alpha+\beta)/2}}{\Gamma \left( \frac{\alpha-\beta+1}{2} \right)} \int_0^\infty (y^2-t^2)^{\left( \frac{\alpha+\beta-1}{2} \right)} y g_1 (y) \, dy. \tag{4.9}
\]

and the generalized Erdelyi-Kober fractional integral operator \([4.1]\) \( K^{(\alpha-\beta+1)/2} \) of order \((\alpha - \beta + 1)/2\) multiplied by a constant.
It is from the definition that \( \hat{f} \in S(R) \) is the Fourier transform of an infinitely differentiable function \( f \) on \( R \) with compact support if and only if

\[
\sigma_{\hat{f}} = \lim_{m \to \infty} \left\| \frac{d^m}{dx^m} \hat{f}(x) \right\|_{L^p(R)}^{1/m}, \quad 1 \leq p < \infty,
\]

(4.10)

where \( \sigma_{\hat{f}} = \sup \left\{ |y| : y \in \text{supp } f \right\} \).

Restricting the Fourier transform only on even functions it is observed that a function \( \hat{f} \in S_e(R) \) is the Fourier cosine transform \((4.8)\) of a function \( h \in S_e \) with compact support if and only if

\[
\sigma_h = \lim_{m \to \infty} \left\| \frac{d^m}{dx^m} f_1(x) \right\|_p^{1/m}.
\]

(4.11)

On the other hand, the Erdelyi-Kober fractional integral operator \( K^{(\alpha - \beta + 1)/2} \) is a bijection in the space of infinitely differentiable functions on \( R_+ \) with compact supports and \( \sigma_h = \sigma_{g_1} \). From \( g_1(y) = y^{-(\alpha + \beta)/2} g(y) \) it is obtained that \( \sigma_g = \sigma_{g_1} \), theorem 4.1 follows now from formula \((4.7)\).

5. Conclusion

1. The Paley-Wiener theorem for the generalized Hankel-Clifford is obtained.
2. The generalized Hankel-Clifford of square integrable functions with compact supports, rapid decreasing functions, infinitely differentiable functions with compact supports, of analytic functions are studied.
3. The range of the generalized Hankel-Clifford transform of compactly supported functions which are either square integrable or infinitely differentiable is characterized.
4. The study leads to application in Mathematical Physics.

Acknowledgement

Author is very much thankful to the referees for their valuable comments and suggestions to improve the paper.

References


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