Polynomial differential systems with hyperbolic algebraic limit cycles

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Abstract. For a given algebraic curve of degree $n$, we exhibit differential systems of degree greater than or equal to $n$, by introducing functions which are solutions of certain partial differential equations. These systems admit precisely the bounded components of the curve as limit cycles.

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1 Introduction

The second part of the sixteenth problem of Hilbert still persists as a research area. It aims to find the maximum number of limit cycles of the differential system:

\begin{align*}
\dot{x} &= \frac{dx}{dt} = P(x,y), \\
\dot{y} &= \frac{dy}{dt} = Q(x,y),
\end{align*}

(1.1)

where $P$ and $Q$ are polynomials.

Several articles and books have been published on the analysis of the existence, number and stability of limit cycles of equation (1.1) (see for instance [5,6,8,9,15,18]).

Generally, the exact analytical expressions of limit cycles for a given differential system are unknown, except in specific cases.

This paper is a contribution in the direction of determining the number of limit cycles and giving their explicit form.

Motivated by some publications [1–4,7,11–14,16], we will exhibit polynomial vector fields, where just by choosing the components of the system satisfying certain conditions, we can conclude directly the number and the explicit form of limit cycles.

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2 Introductory concepts

Let us recall some useful notions.

For $U \in \mathbb{R}[x,y]$, the algebraic curve $U = 0$ is called an invariant curve of the polynomial system (1.1), if for some polynomial $K \in \mathbb{R}[x,y]$, called the cofactor of the algebraic curve, we have

$$P(x,y) \frac{\partial U}{\partial x} + Q(x,y) \frac{\partial U}{\partial y} = KU. \quad (2.1)$$

Simple analysis of equation (2.1) shows that when $\max(\deg P, \deg Q) = n$, the degree of the cofactor $K$ is at most $n - 1$ and that the curve $U = 0$ is formed by trajectories of the system (1.1).

The curve $\Omega = \{(x,y) \in \mathbb{R}^2, U(x,y) = 0\}$ is a non-singular curve of system (1.1), if the equilibrium points of the system that satisfy

$$P(x,y) = 0, \quad Q(x,y) = 0 \quad (2.2)$$

are not contained on the curve $\Omega$.

A limit cycle $\Gamma = \{(x(t), y(t)), t \in [0, T]\}$ is a $T$-periodic solution isolated with respect to all other possible periodic solutions of the system.

A $T$-periodic solution $\Gamma$ is a hyperbolic limit cycle if $\int_0^T \text{div}( \Gamma) dt$ is different from zero.

By using the method of characteristics to solve partial differential equations, we conclude that, the solution of equation

$$a \frac{\partial f}{\partial x} + \beta \frac{\partial f}{\partial y} = 0 \quad (2.3)$$

is

$$f(x,y) = \Phi(\beta x - \alpha y), \quad (2.4)$$

where $\alpha, \beta$ are nonzero reals and $\Phi$ is an arbitrary function.

The solution of the equation

$$a \frac{\partial f}{\partial x} + \beta \frac{\partial f}{\partial y} = \gamma \quad (2.5)$$

is the function $f$ solving the equation

$$\Psi(\beta x - \alpha y, \gamma x - \alpha f) = 0, \quad (2.6)$$

where $\alpha, \beta, \gamma$ are nonzero reals and $\Psi$ is an arbitrary function. In the polynomial case

$$f(x,y) = \frac{\gamma}{\alpha} x + \sum_{k=0}^{n} c_k (\beta x - \alpha y)^k \quad (2.7)$$

or

$$f(x,y) = \frac{\gamma}{\beta} y + \sum_{k=0}^{n} c_k (\beta x - \alpha y)^k \quad (2.8)$$

the solution of the equation

$$x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} = f \quad (2.9)$$
is the function $f$ solving the equation

$$\Psi \left( \frac{x}{f}, \frac{y}{f} \right) = 0.$$  \hfill (2.10)

In the polynomial case it can be taken as

$$f(x, y) = ax + by.$$  \hfill (2.11)

Colin Christopher in his article [7] gives the following theorem.

**Theorem 2.1.** Let $U = 0$ be a non-singular algebraic curve of degree $m$, and $D$ a first degree polynomial, chosen so that the line $D = 0$ lies outside all bounded components of $U = 0$. Choose the constants $a$ and $\beta$ so that $\alpha D_x + \beta D_y \neq 0$, then the polynomial vector field of degree $m$,

$$\begin{align*}
\dot{x} &= \alpha U + DU_y, \\
\dot{y} &= \beta U - DU_x
\end{align*}$$  \hfill (2.12)

has all the bounded components of $U = 0$ as hyperbolic limit cycles. Furthermore, the vector field has no other limit cycles.

Our contribution is a generalization, which consists in introducing polynomial functions to system (2.12) and in the study of the existence of limit cycles.

### 3 The main result

We start by adding a polynomial function of any degree to system (2.12), which becomes,

$$\begin{align*}
\dot{x} &= \alpha U + (ax + by + \Phi(\beta x - \alpha y))U_y, \\
\dot{y} &= \beta U - (ax + by + \Phi(\beta x - \alpha y))U_x
\end{align*}$$  \hfill (3.1)

and we show that system (3.1) has all the bounded components of $U = 0$ as hyperbolic limit cycles if the conditions of Theorem 1 of [7] are satisfied.

**Theorem 3.1.** Let $U = 0$ be a non-singular algebraic curve of degree $m$, and $\Phi$ a polynomial function of degree $n$, chosen so that the curve $ax + by + \Phi(\beta x - \alpha y) = 0$ lies outside all bounded components of $U = 0$. Choose the constants $a$ and $b$ so that $a\alpha + b\beta \neq 0$, then the polynomial vector field of degree $m + n - 1$,

$$\begin{align*}
\dot{x} &= \alpha U + (ax + by + \Phi(\beta x - \alpha y))U_y, \\
\dot{y} &= \beta U - (ax + by + \Phi(\beta x - \alpha y))U_x
\end{align*}$$

has all the bounded components of $U = 0$ as hyperbolic limit cycles.

**Proof.** Let $\Gamma$ be the curve of $U = 0$.

Note that $\Gamma$ is a non-singular curve of system (3.1) and the curve $ax + by + \Phi(\beta x - \alpha y) = 0$ lies outside all bounded components of $\Gamma$.

To show that all the bounded components of $\Gamma$ are hyperbolic limit cycles of system (3.1), we will prove that $\Gamma$ is an invariant curve of the system (3.1), and $\int_0^T \operatorname{div}(\Gamma) dt \neq 0$ (see for instance Perko [17]).
where the cofactor is $K(x, y) = aU_x + \beta U_y$.

ii) $\int_0^T \text{div}(\Gamma) dt$ is nonzero.

To see this, first note that

$$\int_0^T \text{div}(\Gamma) dt = \int_0^T K(x(t), y(t)) dt, \quad (3.2)$$

see for instance Giacomini & Grau [10]. Then one has

$$\int_0^T K(x(t), y(t)) dt = \int_0^T \int_{\Gamma} \frac{\alpha U_x}{(ax + by + \Phi(\beta x - ay))U_x} dy + \int_0^T \int_{\Gamma} \frac{\beta U_y}{(ax + by + \Phi(\beta x - ay))U_y} dx$$

$$= \int_0^T \int_{\Gamma} \frac{\beta}{(ax + by + \Phi(\omega))} dx + \int_0^T \int_{\Gamma} \frac{\alpha}{(ax + by + \Phi(\omega))} dy,$$

Let $\omega = \beta x - ay$. By applying Green’s formula we obtain

$$\int_0^T \frac{\beta}{(ax + by + \Phi(\omega))} dx = \int_0^T \frac{\alpha}{(ax + by + \Phi(\omega))} dy$$

$$= \int_0^T \int_{\text{int}(\Gamma)} \left( \frac{\partial}{\partial y} \left( \frac{\beta}{(ax + by + \Phi(\omega))} \right) + \frac{\partial}{\partial x} \left( \frac{\alpha}{(ax + by + \Phi(\omega))} \right) \right) dxdy$$

$$= \int_0^T \int_{\text{int}(\Gamma)} \left( -\beta \left( b + \frac{\partial \Phi}{\partial y} (\omega) \right) \right) dx dy + \left( -\alpha \left( a + \frac{\partial \Phi}{\partial x} (\omega) \right) \right) dx dy$$

$$= -\int_0^T \int_{\text{int}(\Gamma)} \left( \beta \left( b + \frac{\partial \Phi}{\partial y} (\omega) \right) \right) (ax + by + \Phi(\omega))^2 dxdy + \left( -\alpha \left( a + \frac{\partial \Phi}{\partial x} (\omega) \right) \right) (ax + by + \Phi(\omega))^2 dxdy$$

$$= -\int_0^T \int_{\text{int}(\Gamma)} \left( \beta b + a \left( a + \frac{\partial \Phi}{\partial x} (\omega) \right) \right) dxdy$$

where $\text{int}(\Gamma)$ denotes the interior of $\Gamma$.

As $aa + \beta b \neq 0$, $\int_0^T K(x(t), y(t)) dt$ is nonzero. 

\[\square\]

**Remark 3.2.** When $\Phi(\beta x - ay)$ is constant, we find ourselves in the case of Cristopher’s theorem (i.e. Theorem 2.1).

When $\Phi(\beta x - ay)$ is of first degree, the line $ax + by + c = 0$ in Christopher’s theorem will be replaced by the line $(a + \beta)x + (b - a)y + d = 0$.

**Example 3.3** (Quintic system with exactly one limit cycle). Let $\alpha = 1, \beta = 2, a = 1, b = 2, \Phi(\beta x - ay) = \Phi(2x - y) = (2x - y)^2 + 1$.

The system

$$\begin{align*}
\dot{x} &= x^4 + y^2 - 4y - 3x + 5 + (x + 2y + (2x - y)^2 + 1)(2y - 4), \\
\dot{y} &= 2 \left( x^4 + y^2 - 4y - 3x + 5 \right) - (x + 2y + (2x - y)^2 + 1)(4x^3 - 3) 
\end{align*} \quad (3.3)$$

admits one hyperbolic limit cycle represented by the curve $x^4 + y^2 - 4y - 3x + 5 = 0$. See Figure 3.1.
Remark 3.4. Let us consider the system
\begin{align*}
\dot{x} &= \alpha U + f(x, y)U_y, \\
\dot{y} &= \beta U - f(x, y)U_x,
\end{align*}
where $U$ and $f$ are $C^1$ functions on an open subset $V$ of $\mathbb{R}^2$. To have all the bounded components of $U = 0$ as limit cycles it is necessary that $f$ satisfies the partial differential equation
\begin{equation}
\alpha \frac{\partial f}{\partial x} + \beta \frac{\partial f}{\partial y} = \gamma, \quad \text{where } \gamma \neq 0.
\end{equation}
In the polynomial case $f(x, y) = \frac{\gamma}{\alpha}x + \Phi(\beta x - ay)$ or $f(x, y) = \frac{\gamma}{\beta}y + \Phi(\beta x - ay)$, which are just particular cases of Theorem 3.1.

Example 3.5 (Quintic system with exactly two limit cycles). Let $\alpha = 1$, $\beta = -1$, $\gamma = 3$, $f(x, y) = 3x + (x + y)^2$.

The system
\begin{align*}
\dot{x} &= x^3 - 2xy^2 + 10xy - 15x + y^4 - 10y^3 + 35y^2 - 50y + 30 \\
&\quad + \left( (x + y)^2 + 3x \right) \left( 4y^3 - 30y^2 - 4xy + 10x + 70y - 50 \right), \\
\dot{y} &= 2 \left( x^3 - 2xy^2 + 10xy - 15x + y^4 - 10y^3 + 35y^2 - 50y + 30 \right) \\
&\quad - \left( (x + y)^2 + 3x \right) \left( 3x^2 - 2y^2 + 10y - 15 \right)
\end{align*}
(3.6)
admits two hyperbolic limit cycles represented by the curve $x^3 - 2xy^2 + 10xy - 15x + y^4 - 10y^3 + 35y^2 - 50y + 30 = 0$. See Figure 3.2.
References


