A REPRESENTATION OF BOUNDED COMMUTATIVE BCK-ALGEBRAS

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ABSTRACT. In this note, we prove a representation theorem for bounded commutative BCK-algebras

KEY WORDS AND PHRASES: Bounded commutative BCK-algebra, ideal, prime ideal, quotient BCK-algebras, spectral space

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1. INTRODUCTION

The representation theory of various algebraic structures has been extensively studied. The corresponding representation theory for BCK-algebras remains to be developed. Rousseau and Thaheem [1] proved a representation theorem for a positive implicative BCK-algebra as BCK-algebra of self-mappings which apparently does not possess many algebraic properties. Cornish [2] constructed a bounded implicative BCK-algebra of multipliers corresponding to a bounded implicative BCK-algebra, but no representation of these algebras has been studied there. The purpose of this note is to prove a representation theorem for a bounded commutative BCK-algebra. We essentially prove that a bounded commutative BCK-algebra X is isomorphic to the bounded commutative BCK-algebra X of mappings acting on the associated spectral space of X. Our approach depends on the theory of quotient BCK-algebras as developed by Iséki and Tanaka [3] and the theory of prime ideals of commutative BCK-algebras. Before we develop our results, we recall some technical preliminaries for the sake of completeness. A BCK-algebra is a system (X, *, 0, ≤) (denoted simply by X), satisfying (i) (x * y) * (x * z) ≤ x * y, (ii) x * (x * y) ≤ y, (iii) x ≤ x, (iv) 0 ≤ x, (v) x ≤ y, y ≤ x imply x = y, where x ≤ y if and only if x /∈ 0 for all x, y ∈ X. If X contains an element 1 such that x ≤ 1 for all x ∈ X, then X is said to be bounded. X is said to be commutative if x * y = y * x for all x, y ∈ X, where x ∧ y = y * (y * x). A non-empty set A of a BCK-algebra X is said to be an ideal of X if 0 ∈ A, x, y ∈ A imply y ∈ A, x * z ∈ A implies x ∈ A or y ∈ A. A proper ideal A of a commutative BCK-algebra X is said to be prime if x * y ∈ A implies x ∈ A or y ∈ A. It is well-known that every maximal ideal in a commutative BCK-
algebra is prime (see e.g. [4]). The theory of prime ideals plays an important role in the study of commutative BCK-algebras. For some information about prime ideals, we refer to [5] which contains further references about the theory of prime ideals. A subset $S$ of a commutative BCK-algebra is said to be $\wedge$-closed if $x \wedge y \in S$ whenever $x, y \in S$.

We now state the following theorem known as the prime ideal theorem (see [6, Theorem 2.4] and [5, Corollary 3]).

**Theorem A.** Let $I$ be an ideal and $S$ be a $\wedge$-closed set of a commutative BCK-algebra $X$ such that $S \cap I = \emptyset$. Then there exists a prime ideal $P$ such that $I \subseteq P$ and $P \cap S = \emptyset$.

**Corollary B.** Let $I$ be an ideal of a commutative BCK-algebra $X$ and $a \in X$ such that $a \notin I$. Then there exists a prime ideal $P$ such that $a \notin P$ and $I \subseteq P$.

The above corollary follows from Theorem A by choosing $S = \{a\}$. If a non-trivial commutative BCK-algebra and $I = \{0\}$, then Corollary B ensures the existence of a prime ideal in $X$. We now recall the definition of a quotient BCK-algebra. If $X$ is a BCK-algebra and $A$ is an ideal of $X$, then we define an equivalence relation $\sim$ on $X$ by $x \sim y$ if and only if $x \star y, y \star x \in A$. Let $C_x = \{y \in X : x \star y, y \star x \in A\}$. Let $C_\emptyset = \{y \in X : x \star y, y \star x \in A\}$ denote the equivalence class containing $x \in X$. Then one can see that $C_0 = A$ and $C_x = C_y$ if and only if $x \sim y$. Let $X/A$ denote the set of all equivalence classes $C_x, x \in X$. Then $X/A$ is a BCK-algebra (known as quotient BCK-algebra) with $C_x \star C_y = C_{x \star y}$, and $C_x \leq C_y$ if and only if $x \star y \in A$, and $C_0 = A$ is the zero of $X/A$ (see for instance [3-7]). If $X$ is bounded commutative, then $X/A$ is also bounded commutative with $C_1$ as the unit element.

For the general theory of BCK-algebras and other undefined terminology and notations used here, we refer to Iséki and Tanaka [3-7] and Cornish [8].

**2. A REPRESENTATION THEOREM**

Throughout $X$ denotes a bounded commutative BCK-algebra. Let $\text{Spec}(X)$ denote the set of all prime ideals of $X$, called the spectrum of $X$. It has been shown in [5] that $\text{Spec}(X)$ is a compact topological space referred to as the spectral space associated with $X$. It is well-known that $\bigcap_{P \in \text{Spec}(X)} P = \{0\}$ (see e.g. [8]).

**Definition 2.1.** For any $x \in X$, we define a mapping

$$\widehat{x} : \text{Spec}(X) \to \bigcup_{P \in \text{Spec}(X)} X/P$$

where $\widehat{x}(P)$ denotes the image of $x$ into $X/P$.

It is easy to see that $\widehat{x}(P) = C_0$ if and only if $x \in P$.

We denote by $\hat{X}$, the set of all mappings $\widehat{x}, x \in X$. For any $\widehat{x}, \widehat{y} \in \hat{X}$, we define the following operations on $\hat{X}$:

$$\widehat{x} \star \widehat{y} = (x \star y) \quad \text{and} \quad \widehat{x} \leq \widehat{y} \quad \text{if and only if} \quad \widehat{x} \star \widehat{y} = \widehat{0}.$$ 

These operations are well-defined because of the properties of quotient algebras. Indeed, as $\widehat{x}(P)$ is the canonical image of $x$ in $X/P$, namely the class $C_x$ relative to $P$, and the union $\bigcup_{P \in \text{Spec}(X)} X/P$ is disjoint.

Routine verifications similar to ones for quotient BCK-algebras (see e.g. [3]) lead to the following.

**Proposition 2.2.** $(\hat{X}, \star, \widehat{0})$ is a bounded commutative BCK-algebra.

We now prove the following representation result.

**Theorem 2.3.** The mapping $\phi : x \in X \to \widehat{x} \in \hat{X}$ is an isomorphism.

**Proof.** That $\phi$ is surjective homomorphism follows from the definition (because the mapping $x \in X \to C_x \in X/P$ is the canonical homomorphism). To prove that $\phi$ is injective it is enough to show...
that $\phi(x) = 0$ if and only if $x = 0$. For any $P \in \text{Spec}(X)$, $\phi(x)(P) = 0$ implies that $x \in P$ for all $P \in \text{Spec}(X)$ and hence $x \in \bigcap_{P \in \text{Spec}(X)} P = \{0\}$. Thus $x = 0$. This completes the proof.

We provide an example to explain some essential ideas developed above.

**EXAMPLE 2.4** ([3, p 363]) Let $X = \{0, a, b, 1\}$ be a set. Define a binary operation $*$ on $X$ as in Table 1

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Table 1

The $(X, *, 0)$ is a bounded commutative BCK-algebra with $P = \{0, a\}$ and $Q = \{a, b\}$ as prime ideals (cf Table 2).

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Table 2

Then $\text{Spec}(X) = \{P, Q\}$, $X/P = \{\{0, a\}, \{b, 1\}\}$, $X/Q = \{\{0, b\}, \{a, 1\}\}$, $X/P$, $X/Q$, are disjoint and $\bigcup_{P \in \text{Spec}(X)} X/P$ is the disjoint union as defined above. The rest of the calculations can easily be made to get the representation of $X$ in this case.

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**REFERENCES**


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