OPERATOR SPLITTING METHODS FOR COMPUTATION OF EIGENVALUES OF REGULAR STURM-LIOUVILLE PROBLEMS

İsmail Güzel, Meltem Adıyaman, Sennur Somalı

Abstract. The purpose of this paper is to compute the highest eigenvalues of regular Sturm-Liouville problems with Dirichlet boundary conditions using symmetrical weighted sequential splitting method. The accuracy of the higher eigenvalues of the problem is demonstrated by some classical examples.

1 Introduction

We discuss the computation of higher eigenvalues of regular Sturm-Liouville problem (SLP) in canonical Liouville normal form

\[-y''(t) + q(t)y(t) = \lambda y(t)\]  \hspace{1cm} (1.1)

with Dirichlet boundary conditions

\[y(0) = 0, \quad y(1) = 0\]  \hspace{1cm} (1.2)

for \(q(t) \in C[0,1]\) and \(t \in [0,1]\).

It is well known that, the analysis of an eigenvalue problem as its Liouville form has considerable advantage in terms of theoretical and numerical studies.

An alternative approaches on the same problem SLP with different boundary conditions has been suggested by a number of authors (see, [1, 2, 4, 5]). An annoying aspects of most numerical methods is that the accuracy of the approximation of the \(k^{th}\) eigenvalue decreases as \(k\) increases. In [3, 6, 8, 9, 10, 13], the asymptotic correction techniques are used to produce significant improvement for approximate eigenvalues, but not for all eigenvalues.

In this paper, we apply the symmetrical weighted sequential splitting method given in [11] to compute the higher eigenvalues of SLP (1.1) and (1.2). The operator splitting methods are based on the splitting of variable coefficient system into simpler systems.

2010 Mathematics Subject Classification: 65L15; 34L16;
Keywords: Sturm-Liouville Problem; Operator Splitting Method; Eigenvalues.

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constant and variable coefficient subproblems. So, instead of the original problem, we deal with simpler constant and variable coefficient system using their exact solutions.

In Section 2, we examine the symmetrical weighted sequential splitting method for the approximate determination of the eigenvalues of SLP. In section 3, the asymptotic formula for the error of eigenvalues is given. Our approach to deriving the asymptotic formula is built for \( q(t) = q \). In section 4, some numerical experiments for regular problems are presented by confirming the effectiveness of the suggested approach.

2 Application the Symmetrical Weighted Sequential Splitting Method to Regular SLP

We consider the regular Sturm-Liouville problem (SLP) in canonical Liouville normal form

\[-y''(t) + q(t)y(t) = \lambda y(t), \quad 0 \leq t \leq 1 \quad (2.1)\]

with Dirichlet boundary conditions

\[y(0) = 0, \quad y(1) = 0,\]

where \( q(t) \in C[0,1] \). Equation (2.1) is equivalent with the first order system by \( y' = z \)

\[
\frac{dY(t)}{dt} = A(t)Y(t), \quad 0 \leq t \leq 1, \quad (2.2)
\]

\[C_1Y(0) + C_2Y(1) = 0, \quad (2.3)\]

where

\[Y(t) = \begin{bmatrix} y(t) \\ z(t) \end{bmatrix}, \quad A(t) = \begin{bmatrix} 0 & 1 \\ q(t) - \lambda & 0 \end{bmatrix},\]

\[C_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad C_2 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.\]

The matrix \( A(t) \) is splitted as a sum of \( M \) and \( q(t)N \) such that

\[A(t) = M + q(t)N,\]

where

\[M = \begin{bmatrix} 0 & 1 \\ -\lambda & 0 \end{bmatrix} \quad \text{and} \quad N = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.\]
We consider the partition of the interval $[0, 1]$ as
\[ t_i = ih, \quad i = 0, 1, \ldots, n, \quad h = \frac{1}{n}. \]

The symmetrical weighted sequential splitting of the system on time interval $[t_i, t_{i+1}]$ is defined as in the following algorithm,

\[
\begin{align*}
\frac{dU_1(t)}{dt} &= M U_1(t), \quad U_1(t_i) = Y_{sp,i}, \quad (2.4) \\
\frac{dV_1(t)}{dt} &= q(t)N V_1(t), \quad V_1(t_i) = U_1(t_{i+1}) \quad (2.5)
\end{align*}
\]

and

\[
\begin{align*}
\frac{dU_2(t)}{dt} &= q(t)N U_2(t), \quad U_2(t_i) = Y_{sp,i}, \quad (2.6) \\
\frac{dV_2(t)}{dt} &= M V_2(t), \quad V_2(t_i) = U_2(t_{i+1}), \quad (2.7)
\end{align*}
\]
for $i = 0, 1, \ldots, n - 1$ and $Y_{sp,0}$ is a vector to be determined. The approximate split solution at the point $t_{i+1}$ is defined as
\[ Y_{sp,i+1} = \frac{1}{2} \{ V_1(t_{i+1}) + V_2(t_{i+1}) \}. \]

The exact solution of (2.4) is
\[ U_1(t) = e^{(t-t_i)M}Y_{sp,i}. \]

Since $N$ is nilpotent matrix of index 2 ($N^k = 0, \ k \geq 2$), $q(t)N$ and $(\int^t q(\xi)\ d\xi)N$ are commutative for all $t \in [0, 1]$. Therefore, the solution of the system of differential equation (2.5) is
\[
V_1(t) = e^{w(t)N}U_1(t_{i+1}) \\
= e^{[w(t)-w(t_i)]N} e^{hM}Y_{sp,i},
\]

where $w(t) = \int^t q(\xi) d\xi$, $t \in [t_i, t_{i+1}]$.

By the same consideration, we get
\[
U_2(t) = e^{[w(t)-w(t_i)]N}Y_{sp,i}, \\
V_2(t) = e^{(t-t_i)M}U_2(t_{i+1}) \\
= e^{(t-t_i)M}e^{[w(t_{i+1})-w(t_i)]N}Y_{sp,i}.
\]
Therefore, we can write the approximate split solution as

\[ Y_{sp,i+1} = \frac{1}{2} \left\{ e^{w(t_{i+1})-w(t_i)}N e^{hM} + e^{hM} e^{w(t_{i+1})-w(t_i)}N \right\} Y_{sp,i} \quad (2.8) \]

\[ = \frac{1}{2} \left\{ e^{s_{i+1}N} e^{hM} + e^{hM} e^{s_{i+1}N} \right\} Y_{sp,i}, \quad (2.9) \]

where \( s_{i+1} = s(t_{i+1}) = w(t_{i+1}) - w(t_i) = \int_{t_i}^{t_{i+1}} q(\xi) d\xi, \ i = 0, 1, \ldots, n - 1. \)

Finally, we can write the approximate split solution of (2.2) at \( t_n = 1 \) as

\[ Y(1) \approx Y_{sp,n} = KY_{sp,0}, \]

where \( K \) is a \( 2 \times 2 \) matrix obtained from the recurrence relation (2.8)

\[ K = \frac{1}{2^n} \left\{ \prod_{i=0}^{n-1} [e^{s_{n-i}N} e^{hM} + e^{hM} e^{s_{n-i}N}] \right\}. \]

The solution \( Y_{sp,n} \) will be the solution of (2.2) and (2.3) if and only if

\[ 0 = C_1 Y_{sp,0} + C_2 Y_{sp,n} = C_1 Y_{sp,0} + C_2 K Y_{sp,0} = (C_1 + C_2 K) Y_{sp,0}. \]

For a non-trivial solution \( Y_{sp,0} \), the determinant of \( C_1 + C_2 K \) must be zero. It follows that

\[ Q(\lambda) = \det(C_1 + C_2 K) \quad (2.10) \]

is the approximate characteristic equation of SLP (2.2). So, the approximation to eigenvalue is obtained solving \( Q(\lambda) = 0 \). Now, we task \( e^{hM} \) and \( e^{s(t_i)N} \) to construct the matrix \( K \).

It is apparent that

\[ M^{2j} = (-1)^j \lambda^j I, \quad (2.11) \]

\[ M^{2j+1} = (-1)^j \lambda^j M \quad \text{for} \quad j = 0, 1, \ldots \quad (2.12) \]
Using (2.11) and (2.12), we have
\begin{align*}
e^{tM} &= \left\{ 1 - \frac{\lambda t^2}{2!} + \cdots + (-1)^n \frac{\lambda^n t^{2n}}{(2n)!} + \cdots \right\} I \\
&\quad + \left\{ t - \frac{\lambda t^3}{3!} + \cdots + (-1)^n \frac{\lambda^n t^{2n+1}}{(2n+1)!} + \cdots \right\} M \quad (2.13) \\
&= \cos(\sqrt{\lambda}t)I + \frac{1}{\sqrt{\lambda}} \sin(\sqrt{\lambda}t)M. \quad (2.14)
\end{align*}

Since $N$ is nilpotent matrix of index 2, it is clear that
\begin{equation}
e^{s_{n-i}N} = I + s_{n-i}N. \quad (2.15)
\end{equation}

Using (2.14) and (2.15), we obtain that
\begin{align*}
K &= \frac{1}{2^n} \left\{ \prod_{i=0}^{n-1} \left[ 2e^{hM} + s_{n-i} \left( Ne^{hM} + e^{hM}N \right) \right] \right\},
\end{align*}
where
\begin{equation*}
e^{hM}N + Ne^{hM} = \begin{bmatrix} b(\lambda) & 0 \\ 2a(\lambda) & b(\lambda) \end{bmatrix} = b(\lambda)I + 2a(\lambda)N,
\end{equation*}
and $a(\lambda) = \cos(\sqrt{\lambda}h)$, $b(\lambda) = \frac{1}{\sqrt{\lambda}} \sin(\sqrt{\lambda}h)$.

Then $K$ will be
\begin{align*}
K &= \frac{1}{2^n} \left\{ \prod_{i=0}^{n-1} \left[ 2e^{hM} + s_{n-i} \left[ b(\lambda)I + 2a(\lambda)N \right] \right] \right\}.
\end{align*}

If $q(t) = 0$, then $s_i = 0$. Since $nh = 1$, we have
\begin{align*}
K &= \frac{1}{2^n} \prod_{i=0}^{n-1} 2e^{hM} \\
&= e^M.
\end{align*}

From $\det(C_1 + C_2K) = 0$, we get the characteristic equation of the original SLP
\begin{equation*}
\frac{1}{\sqrt{\lambda}} \sin(\sqrt{\lambda}) = 0
\end{equation*}
and we get the exact eigenvalues of SLP (1.1) and (1.2), \( \lambda_k = k^2 \pi^2, \) \( k = 1, 2, \ldots \) using the proposed method.

Now, we consider the case \( q(t) \) is constant that is \( q(t) = q, \)

\[
    w(t) = \int_{t_i}^{t} q(\xi) \, d\xi \\
    = qt.
\]

From \( t_i = ih, \) we get

\[
    s(t_{i+1}) = w(t_{i+1}) - w(t_i) \\
    = qh.
\]

Then \( K \) will be

\[
    K = \frac{1}{2^n} [2e^{hM} + qh(bI + 2aN)]^n \\
    = \frac{1}{2^n} L^n,
\]

where \( L = \begin{bmatrix} 2a + qhb & 2b \\ -2\lambda b + 2aqh & 2a + qhb \end{bmatrix} \) and \( a(\lambda) := a, \) \( b(\lambda) := b \) for simplicity.

By similarity transformation, it follows that

\[
    L = PDP^{-1},
\]

where

\[
    D = \begin{bmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{bmatrix}, \quad P = \begin{bmatrix} \frac{\sqrt{b(-\lambda b + aqh)}}{-\lambda b + aqh} & -\frac{\sqrt{b(-\lambda b + aqh)}}{-\lambda b + aqh} \\ 1 & 1 \end{bmatrix},
\]

\( \mu_1 \) and \( \mu_2 \) are eigenvalues of \( L, \)

\[
    \mu_{1,2} = 2a + qhb \mp 2\sqrt{b(-\lambda b + aqh)}.
\]

Using \( L^n = PD^nP^{-1}, \) \( (2.10) \) and \( (2.16) \), we have the characteristic function \( Q(\lambda) \) as follows

\[
    Q(\lambda) = \det(C_1 + \frac{1}{2^n} C_2 L^n) \\
    = -\frac{1}{2^{n+1}} \frac{b\sqrt{n}}{\sqrt{aqb - b^2n\lambda}} (\mu_2^n - \mu_1^n),
\]
where
\[
\mu_1^n = \left( \frac{1}{n} \right)^n \left[ 2an + qb + 2\sqrt{bn(-\lambda bn + aq)} \right]^n,
\]
\[
\mu_2^n = \left( \frac{1}{n} \right)^n \left[ 2an + qb - 2\sqrt{bn(-\lambda bn + aq)} \right]^n,
\]
\[a = \cos\left( \frac{\sqrt{\lambda}}{n} \right) \text{ and } b = \frac{1}{\sqrt{\lambda}} \sin\left( \frac{\sqrt{\lambda}}{n} \right).\]

We get limit of the characteristic equation \( Q(\lambda) \) as
\[
\lim_{n \to \infty} Q(\lambda) = \frac{1}{\sqrt{\lambda - q}} \left\{ e^{i\sqrt{\lambda - q}} - e^{-i\sqrt{\lambda - q}} \right\} 
\]
\[
= \frac{1}{\sqrt{\lambda - q}} \sin \sqrt{\lambda - q},
\]
where \( \lambda - q > 0 \). So, we conclude that \( Q(\lambda) \) converges to the characteristic function of the original problem (2.1) for \( q(t) = q \).

3 Asymptotic Behaviour for Eigenvalues of SLP

In order to derive the error estimate \( e_k \), it is necessary to examine in some details of the asymptotic behaviour of error estimate for constant case \( q(t) = q \). Let
\[
|e_k| = |\lambda_k - \lambda_k^{(p+1)}| = |\lambda_k - \left\{ \lambda_k^{(p)} - F(\lambda_k^{(p)}) \right\}|
\]
where \( \lambda_k^{(p)} \) is the \( k^{th} \) approximate eigenvalue to the \( k^{th} \) eigenvalues \( \lambda_k \) of the original problem (2.1) that obtained by Newton method at \( p^{th} \) step, \( F(\lambda) \) is the reduced rational function to \( \frac{Q(\lambda)}{Q'(\lambda)} \). \( Q(\lambda) \) formulated in (2.17) is approximate characteristic equation that obtained from the symmetrical weighted splitting method.

Then, from the convergence of Newton’s iterations, we write
\[
|e_k| = |\lambda_k - \lambda_k^{(p+1)}| \leq |\lambda_k - \lambda_k^{(1)}| \leq |\lambda_k - \lambda_k^{(0)} + F(\lambda_k^{(0)})| \leq |q + F(\lambda_k^{(0)})|,
\]
where $\lambda_k^{(0)} = k^2\pi^2$, $\wedge_k = k^2\pi^2 + q$.

We will discuss the asymptotic behaviour of error formula in two cases.

**Case i:** Let $k = nm + j$, $\lambda_k^{(0)} = (nm + j)^2\pi^2$ and $j = \frac{n}{2}$, $n$ is even number of interval, then

$$a(\lambda) = 0, \quad \sqrt{\lambda} h(\lambda) = (-1)^m, \quad m = 0, 1, \ldots .$$

Using the powers of $\frac{1}{\lambda}$ for asymptotic expansion of the $q + F(\lambda)$, we have

$$|e_k| = |e^2_{\frac{1}{n}(2m+1)}| \leq \frac{|c_1|}{\lambda_k^{(0)}},$$

where $c_1 = (q^2 - \frac{1}{12}q^3) + O(\frac{1}{n})$. We obtain that $e_k \to 0$ as $k \to \infty$ for any constant $q$ and the error is less than 1 if $c_1 < \lambda_k^{(0)}$, i.e,

$$k > \sqrt{\frac{|q^2 - \frac{1}{12}q^3|}{\pi}},$$

for any even $n \geq 2$.

**Case ii:** Let $k = nm + j$, $\lambda_k^{(0)} = (nm + j)^2\pi^2$ and $j \neq \frac{n}{2}$ by using the similar consideration as **Case i**, we get

$$|e_k| = |e_{nm+j}| \leq \frac{|d_1|}{\sqrt{\lambda_k^{(0)}}}, \quad m = 0, 1, \ldots ,$$

where

$$d_1 = \frac{\cos^3(\frac{j}{n}\pi)q^2}{4n \sin(\frac{j}{n}\pi)} + O(\frac{1}{n^2}).$$

It is clear that

$$\left|\frac{\cos^3(\frac{j}{n}\pi)}{n \sin(\frac{j}{n}\pi)}\right| \leq \frac{1}{\pi} \quad \text{for all} \quad j \neq \frac{n}{2}$$

So, we get $d_1 = O(\frac{q^2}{4n^2})$.

Therefore, the error of the eigenvalues is decreasing for $\frac{|d_1|}{\sqrt{\lambda_k^{(0)}}} < 1$, that is,

$$k > \frac{q^2}{4\pi^2}.$$
As a result, from the asymptotic behaviour of the error formulas in Case i and Case ii, we obtain that, for \( m = 0, 1, \ldots, j = 1, 2, \ldots \),

\[
| \lambda_k - \lambda_k^{(p+1)} | = \begin{cases} 
O \left( \frac{1}{k^2} \right), & k = \frac{n}{2} \left( 2m + 1 \right), \quad n : \text{even}, \\
O \left( \frac{1}{k} \right), & k = nm + j, \quad j \neq \frac{n}{2}, 
\end{cases}
\]  
(3.11)

or

\[
| \lambda_k - \lambda_k^{(p+1)} | = \begin{cases} 
O \left( \frac{1}{k^2} \right), & k = \frac{n}{2} \left( 2m + 1 \right), \quad n : \text{even}, \\
O \left( \frac{1}{\sqrt{k}} \right), & k = nm + j, \quad j \neq \frac{n}{2}, 
\end{cases}
\]  
(3.12)

satisfying the conditions (3.7) and (3.10) corresponding to the chosen \( n \).

4 Numerical Results

To illustrate the symmetrical weighted splitting method for Strum-Liouville problems in canonical normal form with Dirichlet boundary conditions, the problem (2.1) is considered with \( q(t) = 5, \quad q(t) = e^t \quad \text{and} \quad q(t) = t^2 \) as considered in [7] and [12]. Throughout tables, the computed \( k^{th} \) approximate eigenvalue for chosen number of intervals \( n \) are denoted by \( \lambda_{k,n} \). Let \( \lambda_k \) be the \( k^{th} \) exact eigenvalue, \( \lambda_{k,n}^{(f)} \) be finite difference approximation for chosen \( n \). For the numerical results, the observed orders are obtained the following formulas

\[
\text{order} = \log \left( \frac{\lambda_{k,n} - \lambda_{k,l}}{\lambda_{r,n} - \lambda_{r,l}} \right) / \log \left( \frac{r}{k} \right),
\]  
(4.1)

\[
\text{order} = \log \left( \frac{\lambda_{s,n} - \lambda_{s,n}}{\lambda_{r,n} - \lambda_{r,n}} \right) / \log \left( \frac{r}{s} \right),
\]  
(4.2)

where \( \lambda_{k,n} \) and \( \lambda_{k,l} \) are the approximate eigenvalues to \( \lambda_k \) for different number of intervals \( n, l \), respectively.

In Table 1 and 2, the errors and the observed orders are given by using the formula (4.1) and (4.1) for even and odd number of intervals \( n \).

Table 3 shows the accuracy of the computed eigenvalues by splitting method \( \lambda_{k,2} \) for \( n = 2 \) by comparing the results of finite difference method \( \lambda_{k,20}^{(f)} \) for \( n = 20 \). It is observed that the results of the presented method for \( n = 2 \) are better than those of the finite difference method for \( n = 20 \).

In Table 4, approximate eigenvalues obtained using presented method and finite difference method of the problem

\[
- y''(t) + e^t y(t) = \lambda y(t), \quad y(0) = y(1) = 0
\]  
(4.3)
are given by comparing the results of the eigenvalues $\lambda^*_k$ given in [12]. In Table 6, approximate eigenvalues obtained using presented method and finite difference method of the problem

$$-y''(t) + t^2 y(t) = \lambda y(t), \quad y(0) = y(1) = 0 \quad (4.4)$$

are introduced by comparing the results of the eigenvalues $\lambda^*_k$ given in [7]. In Table 5 and 7, the greater than 10th eigenvalues are given for problems (4.3) and (4.4).

As a result, the accuracy of the computed eigenvalues by the symmetrical weighted sequential splitting method is better than the finite difference method for the higher eigenvalues without using correction and large $n$.

Table 1: Comparison of the eigenvalues for $n = 2$, $j = 1$ and $n = 6$, $j = 3$ and corresponding orders for $q(t) = 5$ .

| $k$ | $\lambda_{k,2}$ | $|\lambda_{k,2} - \lambda_k|$ | $\lambda_{k,6}$ | $|\lambda_{k,6} - \lambda_k|$ | order |
|-----|----------------|-----------------|----------------|-----------------|-------|
| 15  | 2225.657013825  | 3.97642E-3      | 2225.65805223  | 2.93801E-3      | 1.99442 |
| 39  | 15016.66770447  | 5.89583E-4      | 15016.6678586  | 4.35470E-4      | 1.99927 |
| 81  | 64759.47433883  | 1.36722E-4      | 64759.4743746  | 1.00979E-4      | 1.99983 |
| 219 | 473361.096619   | 1.87049E-5      | 473361.096667  | 1.38147E-5      | 1.99989 |
| 411 | 1667188.445031  | 5.31063E-6      | 1667188.44503  | 3.92250E-6      | 2.00297 |
| 501 | 2477285.574274  | 3.57348E-6      | 2477285.57428  | 2.63983E-6      | 2.00277 |

Table 2: Comparison of the eigenvalues for $n = 3$, $j = 1$ and $n = 5$, $j = 1$ and corresponding orders for $q(t) = 5$ .

| $k$ | $\lambda_{k,3}$ | $|\lambda_{k,3} - \lambda_k|$ | $\lambda_{k,5}$ | $|\lambda_{k,5} - \lambda_k|$ | order |
|-----|----------------|-----------------|----------------|-----------------|-------|
| 16  | 2531.621430695  | 2.70402E-3      | 2531.638123758  | 1.93971E-2      | 1.01109 |
| 31  | 9489.692043667  | 2.21422E-3      | 9489.700596490  | 1.07607E-2      | 1.00718 |
| 61  | 36729.79932009  | 1.34364E-3      | 36729.80364782  | 5.67137E-3      | 1.0042  |
| 121 | 144505.8777071  | 7.33757E-4      | 144505.8809467  | 2.91039E-3      | 1.00188 |
| 301 | 894201.0286518  | 3.08740E-4      | 894201.0295255  | 1.18245E-3      | 1.00083 |
| 436 | 1876177.318445  | 2.15123E-4      | 1876177.319048  | 8.18115E-4      | 1.00063 |
| 541 | 2888650.685889  | 1.74061E-4      | 2888650.686375  | 6.59956E-4      | 1.00051 |
Table 3: Comparison of the errors of eigenvalues obtained using finite difference method and proposed method for $q(t) = 2$.

| $k$ | $\lambda_k$ | $|\lambda_k - \lambda_k^{(f)}|_{20}$ | $|\lambda_k - \lambda_k^{*}|$ |
|-----|-------------|-----------------------------------|-------------------------------|
| 1   | 11.869604401089 | 2.0277E-2 | 9.7745E-2 |
| 3   | 90.826439609804 | 1.6317 | 1.2824E-2 |
| 5   | 248.74011002723 | 12.4255 | 4.6873E-3 |
| 7   | 485.61061565337 | 46.8030 | 2.4017E-3 |
| 9   | 801.43795648823 | 124.5855 | 1.4554E-3 |
| 11  | 1196.2221325318 | 269.0746 | 9.7516E-4 |
| 13  | 1669.9631437841 | 504.7707 | 6.9855E-4 |
| 15  | 2222.6609902451 | 854.9756 | 5.2486E-4 |
| 17  | 2854.3156719148 | 1339.5105 | 4.0871E-4 |
| 19  | 3564.9271887932 | 1972.7765 | 3.2725E-4 |

Table 4: Comparison of the errors of first 10 eigenvalues obtained using finite difference method and proposed method for $q(t) = e^t$.

| $k$ | $n$ | $\lambda_{k,n}$ | $\lambda_k^*$ | $|\lambda_{k,n} - \lambda_k^*|$ | $|\lambda_k^{(f)} - \lambda_k^*|$ |
|-----|-----|-----------------|---------------|-------------------------------|-----------------------------------|
| 1   | 6   | 11.5269698092   | 11.5424       | 1.55302E-2                   | 5.7E-3                            |
| 2   | 4   | 41.1780319772   | 41.1867       | 8.66802E-3                   | 8.13E-2                           |
| 3   | 6   | 90.5364117453   | 90.5404       | 3.98825E-3                   | 4.10E-1                           |
| 4   | 6   | 159.621857116   | 159.6296      | 7.74288E-3                   | 1.2954                            |
| 5   | 2   | 248.454997266   | 248.4569      | 1.90273E-3                   | 3.1544                            |
| 6   | 4   | 357.02185511    | 357.023       | 1.11449E-3                   | 6.5261                            |
| 7   | 2   | 485.327159265   | 485.3281      | 9.40735E-4                   | 12.0593                           |
| 8   | 5   | 633.369784990   | 633.3724      | 2.61501E-3                   | 20.5083                           |
| 9   | 6   | 801.155299106   | 801.1558      | 5.00894E-4                   | 32.7373                           |
| 10  | 4   | 988.677943781   | 988.6783      | 3.56219E-4                   | 49.7023                           |
Table 5: The greater than 10th eigenvalues for \( n = 2, j = 1 \) and \( n = 6, j = 3 \) for \( q(t) = e^t \).

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<td>1.99589</td>
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<td>1618863.58017004</td>
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<td>2597375.63891182</td>
<td>2.20865</td>
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</table>

Table 6: Comparison of the errors of first 10 eigenvalues obtained using finite difference method and proposed method for \( q(t) = t^2 \).

| \( k \) | \( n \) | \( \lambda_{k,n} \) | \( \lambda^*_k \) | \( |\lambda_{k,n} - \lambda^*_k| \) | \( |\lambda^{(f)}_{k,20} - \lambda^*_k| \) |
|-------|------|----------------|----------------|----------------|----------------|
| 1     | 7    | 10.1571617215  | 10.1511640305  | 5.99769E-3     | 2.0291E-2     |
| 2     | 7    | 39.8047902261  | 39.7993930037  | 5.39722E-3     | 3.2365E-1     |
| 3     | 5    | 89.1573512146  | 89.1543424563  | 3.00800E-3     | 1.6316885    |
| 4     | 6    | 158.242156672  | 158.243961707  | 1.80503E-3     | 5.1273118    |
| 5     | 2    | 247.073327812  | 247.071500228  | 1.82722E-3     | 12.425603    |
| 6     | 4    | 355.639012037  | 355.637743806  | 2.68230E-3     | 25.534059    |
| 7     | 2    | 483.943889994  | 483.942959280  | 9.30714E-4     | 46.803153    |
| 8     | 5    | 631.985860302  | 631.987257576  | 1.39727E-3     | 78.868467    |
| 9     | 2    | 799.771254125  | 799.770691532  | 5.62593E-4     | 124.58579    |
| 10    | 7    | 987.293213898  | 987.293288927  | 7.50294E-5     | 186.96079    |
Table 7: The greater than 10\textsuperscript{th} eigenvalues for $n = 2$, $j = 1$ and $n = 6$, $j = 3$ for $q(t) = t^2$.

<table>
<thead>
<tr>
<th>$k$</th>
<th>$\lambda_{k,2}$</th>
<th>$\lambda_{k,6}$</th>
<th>order</th>
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<tbody>
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<td>19986.282244135</td>
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<td>213272.614836339</td>
<td>213272.61483634</td>
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</table>

5 Conclusions

In this paper, the symmetrical weighted sequential splitting method is applied to approximate the eigenvalues of regular Sturm-Liouville problem in Liouville normal form by converting them into a system of differential equation with Dirichlet boundary conditions. To illustrate the method, some classical examples are introduced. From the tables, it is seen that the high order accuracy can be succeeded without increasing the number of intervals $n$ especially for large eigenvalues contrary to finite difference method.

References


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İsmail Güzels
Dokuz Eylül University, The Graduate school of Natural and Applied Science,
İzmir, Turkey.
e-mail:ismailgzel@gmail.com

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Surveys in Mathematics and its Applications 14 (2019), 261 – 275
http://www.utgjiu.ro/math/sma
Meltem Adıyaman  
Dokuz Eylül University, Faculty of Science, Department of Mathematics,  
İzmir, Turkey.  
e-mail: meltem.evrenosoglu@deu.edu.tr

Sennur Somalı  
Dokuz Eylül University, Faculty of Science, Department of Mathematics,  
İzmir, Turkey.  
e-mail: sennur.somali@deu.edu.tr

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