UNIVALENT FUNCTIONS RELATED TO 
q–ANALOGUE OF GENERALIZED M–SERIES 
WITH RESPECT TO k–SYMMETRIC POINTS

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Abstract. In this paper, we introduce subclasses of analytic functions by using q-analogue of generalized M–series and k–symmetric points. Some special coefficient inequalities are also discussed.

1 Introduction

The M–series in [8] is given by:

$$\alpha t^s M_s(z) = \sum_{k=0}^{\infty} \frac{(d_1)_k \cdots (d_t)_k}{(b_1)_k \cdots (b_s)_k} \frac{z^k}{\Gamma(\alpha k + 1)},$$

(1.1)

where $\alpha, z \in \mathbb{C}$, $\text{Re}\{\alpha\} > 0$ and $(d_m)_k, (b_m)_k$ are then well-known Pochhammer symbols.

It is easy to see that by the ratio test the series in (1.1) is convergent for all $z$ if $t \leq s$.

The series in [2] is a convergent series for all $z$ if $t \leq s + \text{Re}\{\alpha\}$. Also it is convergence for $|z| < \alpha^{\alpha}$ if $t = s + \text{Re}\{\alpha\}$.

The q-analogue of Pochhammer symbol is defined by:

$$(\alpha; q)_n = \prod_{k=0}^{n-1} (1 - \alpha q^k), \quad (n \in \mathbb{N}),$$

(1.3)

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and for \( n = 0 \) and \( q \neq 1 \), \((\alpha; q)_0 = 1\). When \( n \to \infty \), we shall assume that \(|q| < 1\), see [2].

Also \( q \)-derivative of a function \( f(z) \) is defined by
\[
(D_q f)(z) = \frac{f(z) - f(qz)}{(1 - q)z}, \quad (z \neq 0, \ q \neq 1),
\]
and
\[
\lim_{q \to 1} D_q f(z) = f'(z).
\]

By using (1.4), we conclude that:
\[
(D^n_q f)(x) = q^n (D^n_q f)(\frac{x}{q^n}),
\]
\[
D^n_q z^\lambda = \frac{\Gamma_q(\lambda + 1)}{\Gamma_q(\lambda - n + 1)} z^{\lambda - n}, \quad (\text{Re}\{\lambda\} + 1 \geq 0).
\]

Indeed
\[
\Gamma_q(z + 1) = \frac{1 - q^z}{1 - z} \Gamma_q(z).
\]

Also
\[
\beta_q(x, y) = \int_0^1 t^{x-1} (\frac{tq; q}{tq^y; q})_\infty dq(t) = \frac{\Gamma_q(x)\Gamma_q(y)}{\Gamma_q(x + y)},
\]
where \( \text{Re}\{x\} > 0, \text{Re}\{y\} > 0 \) and \( \beta_q(x, y) \) and \( \Gamma_q(w) \) are the \( q \)-analogue of the beta function and \( q \)-gamma function respectively.

Now we consider the \( q \)-analogue of generalized \( M \)-series as follows:
\[
\alpha^\beta M^\beta_q(z; q) = \sum_{k=0}^{\infty} \frac{(d_1; q)_k \cdot (d_i; q)_k}{(b_1; q)_k \cdots (b_i; q)_k (q; q)_k} \frac{z^k}{\Gamma_q(\alpha k + \beta)},
\]
where \( \alpha, \beta \in \mathbb{C}, \text{Re}\{\alpha\} > 0, |q| < 1, \ (\gamma; q)_k \) is the \( q \)-analogue of Pochhammer symbol and \( \Gamma_q(w) \) is the \( q \)-gamma function.

By applying the convergent conditions of the well-known Fox-Wright generalized hypergeometric function and generalized \( H \)-function, the function \( \alpha^\beta M^\beta_q(z; q) \) is convergent, see [3].

Some special cases of \( \alpha^\beta M^\beta_q(z; q) \) are:

(i) The \( q \)-Mittag-Leffler function [5].
(ii) The generalized \( q \)-Mittag-Leffler function [10].

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(iii) The $q$-generalized $M$-series as a special case of the $q$-Wright generalized hypergeometric function [6].

**Definition 1.** Let $A$ denote the class of functions $f(z)$ of the type:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k,$$  \hspace{1cm} (1.11)

which are analytic in the open unit disk:

$$U = \{ z \in \mathbb{C} : |z| < 1 \}.$$

**Definition 2.** the Hadamard product (convolution) for functions $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ and $g(z) = z + \sum_{k=2}^{\infty} b_k z^k$ belong to $A$ denoted by $f \ast g$ in defined as follows:

$$(f \ast g)(z) = z + \sum_{k=2}^{\infty} a_k b_k = (g \ast f)(z).$$  \hspace{1cm} (1.12)

**Definition 3.** A function $f(z) \in A$ is in the class $X_n(\theta)$ if

$$\text{Re} \left\{ \frac{z(f \ast F)' + (f \ast F)}{H_n(z)} \right\} < \theta,$$  \hspace{1cm} (1.13)

where $\theta > 1$, $n \geq 1$ is a fixed positive integer,

$$F(z) = \left[ 1 - \frac{(1-d_1)(1-d_t)}{(1-b_1)(1-b_t)\Gamma(\alpha + \beta)} \right] z - \frac{1}{\Gamma_q(\beta)} + q^\beta \mathcal{M}_q^\beta(z;q),$$  \hspace{1cm} (1.14)

and

$$H_n(z) = \frac{1}{n} \sum_{v=0}^{n-1} E^{-v} (f \ast F)(E^v z), \hspace{1cm} (E^n = 1, \ z \in U).$$  \hspace{1cm} (1.15)

Further, a function $f(z) \in A$ is in the class $Y_n(\theta)$, if and only if $zf'(z) \in X_n(\theta)$.

From (1.11), (1.14) and (1.12) with a simple calculation, we get:

$$(f \ast F)(z) = z + \sum_{k=2}^{\infty} \frac{(d_1; q)_k \ldots (d_t; q)_k}{(b_1; q)_k \ldots (b_t; q)_k (q; q)_k \Gamma_q(\alpha k + \beta)} a_k z^k.$$  \hspace{1cm} (1.16)

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2 Main results

In this section, we shall obtain some coefficient bounds for functions in $X_n(\theta)$ and $Y_n(\theta)$ and their subclasses positive coefficients.

Note that other subclasses of analytic functions with respect to $n$-symmetric points have been studied by many authors, see [4, 7] and [11].

Theorem 4. Let $\theta > 2$. If $f(z) \in A$ satisfies:

\[
\sum_{k=1}^{\infty} \frac{(nk+2)(d_1;q)_{nk+1} \cdots (d_i;q)_{nk+1}}{(b_1;q)_{nk+1} \cdots (b_i;q)_{nk+1}(q;q)_{nk+1}\Gamma(\alpha(nk+1)+\beta)}
\]

we get

\[
\sum_{k=2}^{\infty} \frac{(nk+2)(d_1;q)_{nk+1} \cdots (d_i;q)_{nk+1}}{(b_1;q)_{nk+1} \cdots (b_i;q)_{nk+1}(q;q)_{nk+1}\Gamma(\alpha(nk+1)+\beta)} - 2\theta z^{-1}
\]

then $f \in X_n(\theta)$.

Proof. Suppose that $\theta > 2$ and $f(z) \in A$, it is sufficient to show that:

\[
\frac{|zf(F)' + f(F)|}{f_n(z)} < \left| \frac{z(F(F)' + f(F) - 2\theta}{f_n(z)} \right|, \quad (z \in \mathbb{U}).
\]

By putting

\[
W = |z(F(F)' + f(F)| - |z(F(F)' + f(F) - 2\theta f_n(z)|,
\]

and

\[
H_n(z) = z + \frac{1}{n} \sum_{k=2}^{\infty} \frac{(d_1;q)_k \cdots (d_i;q)_k}{(b_1;q)_k \cdots (b_i;q)_k(q;q)_k\Gamma(\alpha k + \beta)} \sum_{v=0}^{n-1} E_v^{(k-1)} z^n,
\]

we get

\[
W = \left| 2z + \sum_{k=2}^{\infty} \frac{(d_1;q)_k \cdots (d_i;q)_k}{(b_1;q)_k \cdots (b_i;q)_k(q;q)_k\Gamma(\alpha k + \beta)} a_k z^k \right|
\]

\[
- 2z + \sum_{k=2}^{\infty} \frac{(d_1;q)_k \cdots (d_i;q)_k a_k z^k}{(b_1;q)_k \cdots (b_i;q)_k(q;q)_k\Gamma(\alpha k + \beta)}
\]

\[
- 2\theta z - 2\theta \sum_{k=2}^{\infty} \frac{(d_1;q)_k \cdots (d_i;q)_k a_k b_k z^k}{(d_1;q)_k \cdots (d_i;q)_k(q;q)_k\Gamma(\alpha k + \beta)}
\]
where
\[
b_k = \frac{1}{n} \sum_{v=0}^{n-1} E^v(k-1), \quad (E^n = 1). \tag{2.3}
\]

Hence for \(|z| = r < 1\), we have:
\[
W \leq 2r + \sum_{k=2}^{\infty} \frac{(k + 1)(d_1; q)_k \cdots (d_i; q)_k a_k r^k}{(b_1; q)_k \cdots (b_s; q)_k(q; q)_k \Gamma(\alpha k + \beta)}
\]
\[
- \left\{ 2(\theta - 1)r - \sum_{k=2}^{\infty} \frac{(k + 1)(d_1; q)_k \cdots (d_i; q)_k}{(b_1; q)_k \cdots (b_s; q)_k(q; q)_k \Gamma(\alpha k + \beta)} - 2\theta b_k \right\} |a_k|^k \right\}
\]
\[
< \left\{ \sum_{k=2}^{\infty} \frac{(k + 1)(d_1; q)_k \cdots (d_i; q)_k}{(b_1; q)_k \cdots (b_s; q)_k(q; q)_k \Gamma(\alpha k + \beta)}
\]
\[
+ \left. \frac{(k + 1)(d_1; q)_k \cdots (d_i; q)_k}{(b_1; q)_k \cdots (b_s; q)_k(q; q)_k \Gamma(\alpha k + \beta)} - 2\theta b_k \right] |a_k| - 2(\theta - 2) \right\} r.
\]

From (2.3) we know
\[
b_k = \begin{cases} 
0 & , \ k \neq mn + 1 \\
1 & , \ k = mn + 1.
\end{cases} \tag{2.4}
\]

So we get
\[
W < \left\{ \sum_{k=1}^{\infty} \frac{(nk + 2)(d_1; q)_{nk+1} \cdots (d_i; q)_{nk+1}}{(b_1; q)_{nk+1} \cdots (b_s; q)_{nk+1} q(q)_{nk+1} \Gamma(\alpha (nk + 1) + \beta)}
\]
\[
+ \left. \frac{(nk + 2)(d_1; q)_{nk+1} \cdots (d_i; q)_{nk+1}}{(b_1; q)_{nk+1} \cdots (b_s; q)_{nk+1} q(q)_{nk+1} \Gamma(\alpha (nk + 1) + \beta)} - 2\theta \right] \left|a_{nk+1}\right| 
\]
\[
+ \sum_{\substack{k=2 \ \text{to} \ \infty}}^{\infty} \frac{2(k + 1)(d_1; q)_k \cdots (d_i; q)_k}{(b_1; q)_k \cdots (b_s; q)_k(q; q)_k \Gamma(\alpha k + \beta)} |a_k| - 2(\theta - 2) \right\} r.
\]

From (2.1) we know that \(W < 0\). Thus we get then required result.

By definition of \(Y_n(\theta)\) we obtain the following corollary.
Corollary 5. If \( \theta > 2 \) and \( (f * F)(z) \) is defined by (1.16) satisfies

\[
\sum_{k=0}^{\infty} (nk + 2) \left[ \frac{(nk + 2)(d_1; q)_{nk+1} \cdots (d_t; q)_{nk+1}}{b_1; q)_{nk+1} \cdots (b_s; q)_{nk+1} (q; q)_{nk+1} \Gamma(\alpha(nk + 1) + \beta)} \right] |a_{nk+1}| \leq \theta
\]

then \( f(z) \in Y_n(\theta) \).

Now, we define two subclasses of \( X_n(\theta) \) and \( Y_n(\theta) \) as follow:

\[
X_n^+(\theta) = \left\{ f \in X_n(\theta) : \text{The coefficients of } f * F \text{ are non-negative} \right\},
\]

\[
Y_n^+(\theta) = \left\{ f \in Y_n(\theta) : \text{The coefficients of } f * F \text{ are non-negative} \right\}.
\]

Theorem 6. Let \( n \geq 2 \) and \( 2 < \theta \leq n + 1 \), then \( f(z) \in X_n^+(\theta) \) if and only if

\[
\sum_{k=2}^{\infty} \frac{(k+1)(d_1; q)_{k} \cdots (d_t; q)_{k}}{(b_1; q)_{k} \cdots (b_s; q)_{k} (q; q)_{k} \Gamma(\alpha k + \beta)} a_k - \theta \sum_{m=1}^{\infty} \frac{(d_1; q)_{mn+1} \cdots (d_t; q)_{mn+1} a_{mn+1}}{(b_1; q)_{mn+1} \cdots (b_s; q)_{mn+1} (q; q)_{mn+1} \Gamma(\alpha mn + 1) + \beta)} \leq 2(\theta - 2).
\]

Proof. According to Theorem 4, we need only to prove the necessary. Since \( f(z) \in X_n^+(\theta) \), then

\[
\frac{(d_1; q)_{k} \cdots (d_t; q)_{k}}{(b_1; q)_{k} \cdots (b_s; q)_{k} (q; q)_{k} \Gamma(\alpha k + \beta)} a_k \geq 0, \quad (k \geq 2).
\]

Also

\[
\text{Re}\left\{ \frac{z(f * F) + (f * F)}{H_n(z)} \right\} < \theta,
\]

or equivalently

\[
\left| \frac{z(f * F) + (f * F)}{H_n(z)} \right| < \left| \frac{z(f * F) + (f * F)}{H_n(z)} \right|,
\]

or

\[
|z(f * F) + (f * F)| < |z(f * F) + (f * F)| - 2\theta H_n(z).
\]

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Hence
\[
\left| z + \sum_{k=2}^{\infty} \frac{k(d_1:q)_k \cdots (d_i:q)_k}{(b_1:q)_k \cdots (b_r:q)_k \Gamma_q(\alpha k + \beta)} a_k z^k \right|
\]
\[
+ z + \sum_{k=2}^{\infty} \frac{(d_1:q)_k \cdots (d_i:q)_k}{(b_1:q)_k \cdots (q:q)_k \Gamma_q(\alpha k + \beta)} a_k z^k
\]
\[
< \left| z + \sum_{k=2}^{\infty} \frac{k(d_1:q)_k \cdots (d_i:q)_k}{(b_1:q)_k \cdots (q:q)_k \Gamma_q(\alpha k + \beta)} a_k z^k \right|
\]
\[
+ z + \sum_{k=2}^{\infty} \frac{(d_1:q)_k \cdots (d_i:q)_k}{(b_1:q)_k \cdots (q:q)_k \Gamma_q(\alpha k + \beta)} + 2 \theta a_k z^k
\]
\[
- 2 \theta \sum_{m=1}^{\infty} \frac{(d_1:q)_{mn+1} \cdots (d_i:q)_{mn+1} a_{mn+1}}{(b_1:q)_{mn+1} \cdots (q:q)_{mn+1} \Gamma_q(\alpha mn + \beta)} z^{mn+1} a_k.
\]
Since
\[
\frac{(d_1:q)_k \cdots (d_i:q)_k}{(b_1:q)_k \cdots (q:q)_k \Gamma_q(\alpha k + \beta)} a_k \geq 0,
\]
for \( k \geq 2 \) and \( \theta > 2 \), by setting \( z \rightarrow 1^− \), we get:
\[
2 + \sum_{k=2}^{\infty} (k+1) \frac{(d_1:q)_k \cdots (d_i:q)_k}{(b_1:q)_k \cdots (q:q)_k \Gamma_q(\alpha k + \beta)} a_k \leq 2 \theta - 2
\]
\[
+ 2 \theta \sum_{m=1}^{\infty} \frac{(d_1:q)_{mn+1} \cdots (d_i:q)_{mn+1} a_{mn+1}}{(b_1:q)_{mn+1} \cdots (q:q)_{mn+1} \Gamma_q(\alpha mn + \beta)} a_{mn+1}
\]
\[
- \sum_{k=2}^{\infty} (k+1) \frac{(d_1:q)_k \cdots (d_i:q)_k}{(b_1:q)_k \cdots (q:q)_k \Gamma_q(\alpha k + \beta)} a_k,
\]
or
\[
\sum_{k=2}^{\infty} (k+1) \frac{(d_1:q)_k \cdots (d_i:q)_k}{(b_1:q)_k \cdots (q:q)_k \Gamma_q(\alpha k + \beta)} a_k
\]
\[
- \theta \sum_{m=1}^{\infty} \frac{(d_1:q)_{mn+1} \cdots (d_i:q)_{mn+1} a_{mn+1}}{(b_1:q)_{mn+1} \cdots (q:q)_{mn+1} \Gamma_q(\alpha mn + \beta)} \leq 2(\theta - 2).
\]

Similarly, we have the following theorem for the class \( Y_n(\theta) \).

**Corollary 7.** Let \( n \geq 2 \), \( 2 < \theta \leq n + 1 \) and \( f(z) \in \mathcal{A} \), then \( f(z) \) is in the class \( Y_n(\theta) \) if and only if
\[
\sum_{k=2}^{\infty} (k+1)^2 \frac{(d_1:q)_k \cdots (d_i:q)_k}{(b_1:q)_k \cdots (q:q)_k \Gamma_q(\alpha k + \beta)} a_k
\]
\[ -\theta \sum_{m=1}^{\infty} (mn + 1) \frac{(d_1; q)_{mn+1} \cdots (d_t; q)_{mn+1} a_{mn+1}}{(b_1; q)_{mn+1} \cdots (q; q)_{mn+1} \Gamma_q(\alpha mn + 1 + \beta)} \leq 2(\theta - 2). \]

References


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Univalent functions related to $q$–analogue of generalized $M$–series

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